Math 120 - Autumn 2016
Final Exam
December 10, 2016

Name: ____________________________  Student ID no.: ____________
Signature: ____________________________  Section: ____________

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- This exam consists of SEVEN problems on EIGHT pages, including this cover sheet.
- Show all work for full credit.
- You may use a TI-30X IIS calculator during this exam. Other calculators and electronic device are not permitted.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you write on the back of the page, please indicate that you have done so!
- **Draw a box** around your final answer to each problem.
- You may use one hand-written double-sided 8.5” by 11” page of notes.
- You have 170 minutes to complete the exam.
1. [5 points per part] For parts (a) through (c), consider the following multipart function:

\[ f(x) = \begin{cases} 
1 + \sqrt{9 - x^2} & \text{if } -3 \leq x < 0 \\
2x & \text{if } 0 \leq x \leq 2 \\
1 & \text{if } 2 < x \leq 4 
\end{cases} \]

(a) Sketch a graph of \( f(x) \).

(b) Find all values of \( x \) such that \( f(x) = -2x + 2.2 \).

- For \(-3 \leq x < 0\):
  \[ 1 + \sqrt{9 - x^2} = -2x + 2.2 \]
  \[ \sqrt{9 - x^2} = -2x + 1.2 \]
  \[ 9 - x^2 = (4x - 1.2)^2 \]
  \[ 0 = 5x^2 - 4.8x - 7.56 \]
  \[ x = \frac{4.8 \pm \sqrt{4.8^2 + 5 \times 7.56}}{10} \]
  \[ x = 1.04 \text{ or } -0.84 \]

  \[ 0 \leq x \leq 2: \quad 2x = -2x + 2.2 \]
  \[ 4x = 2.2 \]
  \[ x = 0.55 \]

  \[ 2 < x \leq 4: \quad 1 = -2x + 2.2 \]
  \[ 1, 2 = 2x \]
  \[ x = 0.5 \]

  Two solutions

(c) Let \( g(x) \) be the function found by taking the graph of \( f(x) \) and shifting it 1 unit left.

Write the multipart rule for \( g(x) \).

- Replace \( x \) with \( x + 1 \).
- Shift intervals left.

\[ g(x) = \begin{cases} 
1 + \sqrt{9 - (x+1)^2} & \text{if } -4 \leq x < -1 \\
2(x+1) & \text{if } -1 \leq x \leq 1 \\
1 & \text{if } 1 < x \leq 3 
\end{cases} \]
2. (a) [4 points] A mysterious red dot is moving through the $xy$-plane at a constant speed. 
At time $t = 0$, it starts at $(-1, -1)$. It moves in a straight line towards the point $(31, 13)$, reaching it in 10 seconds.
Write parametric equations for the red dot’s coordinates after $t$ seconds.

\[
\begin{align*}
\Delta x &= 32 \\
\Delta y &= 14 \\
\Delta t &= 10
\end{align*}
\]

\[
\begin{align*}
x &= -1 + 3.2t \\
y &= -1 + 1.4t
\end{align*}
\]

(b) [5 points] Fungo is also moving in the $xy$-plane. At time $t = 0$, he starts at $(2, 6)$.
Fungo runs in a straight line towards $(8, -2)$ at a speed of 2 units per second.
Write parametric equations for Fungo’s coordinates after $t$ seconds.

\[
\begin{align*}
\Delta x &= 6 \\
\Delta y &= -8 \\
\Delta t &= \frac{\text{dist}}{\text{speed}} = \frac{\sqrt{6^2 + 8^2}}{2} = 5
\end{align*}
\]

\[
\begin{align*}
x &= 2 + 1.2t \\
y &= 6 - 1.6t
\end{align*}
\]

(c) [7 points] When is Fungo closest to the red dot?

\[
d_{\text{dist}} = \sqrt{((-1 + 3.2t) - (2 + 1.2t))^2 + ((-1 + 1.4t) - (6 - 1.6t))^2}
\]

\[
= \sqrt{(-3 + 2t)^2 + (-7 + 3t)^2}
\]

\[
= \sqrt{9 - 12t + 4t^2 + 49 - 42t + 9t^2}
\]

\[
= \sqrt{13t^2 - 54t + 58}
\]

\[
\text{Minimum when } t = h = \frac{-b}{2a} = \frac{-54}{2 \cdot 13} = \frac{54}{26} \approx 2.077 \text{ sec.}
\]
3. The population of Threeattle triples every ten years.

Four years from now, there will be 10,000 more people in Threeattle than there are today.

(a) [6 points] Write a function \( f(x) \) for the population of Threeattle \( x \) years from today.

\[
f(x) = A_0 b^x
\]

\[
b^{10} = 3 \quad \rightarrow \quad b = 3^{\frac{1}{10}}
\]

\[
f(4) = f(0) + 10000
\]

\[
A_0 b^4 = A_0 + 10000
\]

\[
A_0 (b^4 - 1) = 10000
\]

\[
A_0 = \frac{10000}{b^4 - 1} = \frac{10000}{3^{\frac{4}{10}} - 1} = 18121
\]

(b) [6 points] Compute the inverse of the function you found in part (a).

\[
y = 18121 \left(3^{\frac{1}{10}}\right)^x
\]

\[
y = \frac{18121}{18121}
\]

\[
\ln \left(\frac{y}{18121}\right) = \ln \left(\left(3^{\frac{1}{10}}\right)^x\right)
\]

\[
\ln \left(\frac{y}{18121}\right) = x \ln \left(3^{\frac{1}{10}}\right)
\]

\[
x = \frac{\ln \left(\frac{y}{18121}\right)}{\ln \left(3^{\frac{1}{10}}\right)}
\]

\[
f^{-1}(y) = \frac{\ln \left(\frac{y}{18121}\right)}{\ln \left(3^{\frac{1}{10}}\right)}
\]

(c) [2 points] In one sentence, explain the meaning of the inverse function you found.

\( f^{-1}(y) \) is the year when the population will reach \( y \).
4. **[14 points]** Angelica and Eliza are standing 100 meters apart, and Peggy is standing exactly halfway between them. Somewhere else in between them is a tree.

From where they stand, Angelica, Eliza, and Peggy all measure the angle of elevation of the top of the tree above the ground.

Angelica measures it to be $32^\circ$. Eliza measures it to be $27^\circ$.

What measurement does Peggy get?

\[
\tan(32^\circ) = \frac{y}{50-x} \rightarrow y = (50-x)\tan(32^\circ)
\]

\[
\tan(27^\circ) = \frac{y}{50+x} \rightarrow y = (50+x)\tan(27^\circ)
\]

\[
(50-x)\tan(32^\circ) = (50+x)\tan(27^\circ)
\]

\[
50\tan(32^\circ) - 50\tan(27^\circ) = x\tan(32^\circ) + x\tan(27^\circ)
\]

\[
x = \frac{50(\tan(32^\circ) - \tan(27^\circ))}{\tan(32^\circ) + \tan(27^\circ)} \approx 5.0839
\]

\[
y = (50+x)\tan(27^\circ) \approx 28.0667
\]

\[
\Theta = \tan^{-1}\left(\frac{y}{x}\right) \approx 79.7329^\circ
\]
5. \( f(x) \) is a linear-to-linear rational function whose graph has a horizontal asymptote of \( y = 5 \) and passes through the points \((1, -10)\) and \((2, -35)\).

(a) [7 points] Write a formula for \( f(x) \).

\[
\begin{align*}
\frac{a}{x + \frac{b}{x + d}} &= -10 \\
-10 &= \frac{a + b}{1 + d} \\
-35 &= \frac{2a + b}{2 + d}
\end{align*}
\]

\[
\begin{align*}
a &= 5 \\
b &= -10 + 26 - 5 \\
d &= -2.6
\end{align*}
\]

\[
f(x) = \frac{5x + 11}{x - 2.6}
\]

(b) [2 points] What is the domain of \( f(x) \)?

Everything but 2.6:

\((-\infty, 2.6) \cup (2.6, \infty)\)

(c) [6 points] Let \( g(x) = f(f(x)) \). Find the asymptotes of \( g(x) \).

\[
g(x) = f\left( f(x) \right) = \frac{5 \left( \frac{5x + 11}{x - 2.6} \right) + 11}{\left( \frac{5x + 11}{x - 2.6} \right) - 2.6}
\]

\[
g(x) = \frac{36x + 26.4}{2.4x + 17.76} = \frac{15x + 11}{x + 7.4}
\]

Asymptotes:

\( y = 15 \)

\( x = -7.4 \)
6. [14 points] The temperature in Meereen is a sinusoidal function of time. The temperature will decrease for the next 5 years until it reaches a minimum of 20°. Then the temperature will climb until 21 years from now, when it reaches a maximum of 80°. Dany is unhappy when the temperature in Meereen is below 70°. Over the next 100 years, for how long will she be unhappy?

\[ f(x) = 30 \sin \left( \frac{2\pi}{32} (x-13) \right) + 50 \]

**First set of solutions:**

\[ x = 13 + \frac{32}{2\pi} \sin^{-1} \left( \frac{2}{3} \right) = 16.716 \]  
\[ 26 + \frac{32}{2\pi} \sin^{-1} \left( \frac{2}{3} \right) = 25.283 \]  
\[ 57 + \frac{32}{2\pi} \sin^{-1} \left( \frac{2}{3} \right) = 80.716 \]  
\[ 89 + \frac{32}{2\pi} \sin^{-1} \left( \frac{2}{3} \right) = 89.283 \]

**Total time:**

\[ (16.716-0) + (48.716-25.283) + (80.716-57.283) + (100-89.283) = 74.299 \text{ years} \]
7. (a) [12 points] Essun is running 3 meters per second clockwise around a circular track.

From her starting point, it takes her 9 seconds to reach the northernmost point of the track, and then an additional 13 seconds to reach the easternmost point of the track.

After 2 minutes, how far east is Essun from the westernmost point of the track?

\[
\begin{align*}
v &= 3 \\
\omega &= \frac{\pi}{\frac{3}{13}} = \frac{13\pi}{26} \\
r &= \frac{v}{\omega} = \frac{78}{13} \\
\theta_0 &= 0 \\
\theta &= \frac{\pi}{2} + \frac{9\pi}{26} = \frac{22\pi}{26} = \frac{11\pi}{13} \\
x &= 0 + \frac{78}{13} \cos\left(\frac{11\pi}{13} - \frac{\pi}{13}\right) \approx 18.584 \\
\text{Westernmost point is at } (-\frac{78}{13}, 0), \text{ so this is } 18.584 + \frac{78}{13} \approx 43.412 \text{ m east}
\end{align*}
\]

(b) [0 points] You’re done! Please check your work, then enjoy this celebratory maze.

Help Essun get to the exit. Unfortunately, after spending so much time running clockwise, she has forgotten how to turn left.