Math 120 C and E - Autumn 2004
Mid-Term Exam Number Two Solutions
November 18, 2004

1. *Shirley knows that the more weight she loads on her bicycle, the slower she will have to ride up Morgan's Hill. If she carries no load, she can ride 12 feet per second up the hill. With a load of 50 pounds, her speed drops to 6 feet per second. With a load of 100 pounds, her speed is 5 feet per second.*

   Assuming her speed is a linear-to-linear rational function of her load, what will her speed be at a load of 25 pounds?

   Let $s(x)$ be the Shirley's speed with a load of x pounds. Then, for some A, B, and C,

   $$s(x) = \frac{Ax + B}{x + C}$$

   We know

   $$s(0) = 12$$

   $$s(50) = 6$$

   and

   $$s(100) = 5.$$

   So we have the equations

   $$\frac{B}{C} = 12 \text{ or } B = 12C$$

   $$\frac{50A + 12C}{50 + C} = 6 \text{ or } 50A + 12C = 300 + 6C$$

   $$\frac{100A + 12C}{100 + C} = 5 \text{ or } 100A + 12C = 500 + 5C$$

   Doubling the first of the last two equations and subtracting the last we get

   $$12C = 100 + 7C$$

   or

   $$C = 20$$

   from which we find

   $$B = 12(C) = 240$$

   and

   $$A = \frac{300 - 6C}{50} = 3.6$$

   so that

   $$s(x) = \frac{3.6x + 240}{x + 20}$$

   and her speed with a load of 25 pounds is

   $$s(25) = 7.333333... \text{ ft/sec.}$$

2. *Let $f(x)$ be defined by*

   $$f(x) = \frac{3x + 5}{x - 6}.$$

   Find $f^{-1}(x)$.

   Let

   $$y = \frac{3x + 5}{x - 6}$$

Then we solve for x:

$$xy - 6y = 3x + 5$$
$$xy - 3x = 5 + 6y$$
$$x(y-3) = 5 + 6y$$
$$x = \frac{5+6y}{y-3}.$$

So

$$f^{-1}(x) = \frac{5+6x}{x-3}$$

3. *A scientist in the arctic noticed one day that the thickness of the ice at a certain point on the sea was varying sinusoidally. She found that the ice reached its minimum thickness of 90cm at 1:30 PM. It then grew to its maximum thickness of 180cm at 4:00 PM.*

   How thick was the ice at 6:15 PM?

   Let $f(t)$ be the thickness of the ice t hours after noon. Then

   $$f(1.5) = 90 \text{ and } f(4) = 180$$

   The function f is sinusoidal, so

   $$f(t) = A\sin\left(\frac{2\pi}{B}(t-C)\right) + D$$

   for constants A, B, C and D.

   From this given information, we have

   $$A = \frac{180-90}{2} = 45$$
   $$D = \frac{90+180}{2} = 135$$
   $$B = 2(4-1.5) = 2(2.5) = 5$$
   $$C = 4 - \frac{B}{4} = 4 - \frac{5}{4} = 2.75$$

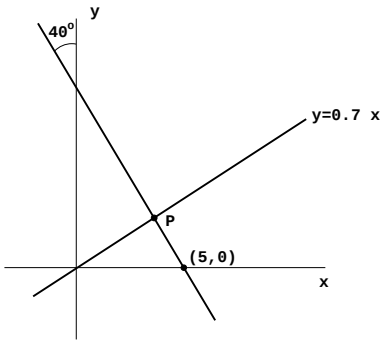
   Hence,

   $$f(t) = 45\sin\left(\frac{2\pi}{5}(t-2.75)\right) + 135$$

   and the thickness at 6:15 PM is

   $$f(6.25) = 45\sin\left(\frac{2\pi}{5}(6.25-2.75)\right) + 135 = 92.202456766718089254 76...\text{cm}.$$

4. *Find the coordinates of the point P in the figure below.*



There are two lines (besides the axes) in the figure. One is $y = 0.7x$. Suppose the other has y-intercept b. Then, from the figure

$$\frac{5}{b} = \tan 40^\circ$$

while the slope of the line is

$$m = -\frac{b}{5}$$

so

$$m = -\frac{1}{\tan 40^\circ}$$

Thus this line has equation

$$y = -\frac{1}{\tan 40^\circ}(x - 5)$$

The point P is the intersection of the two lines:

$$0.7x = -\frac{1}{\tan 40^\circ}(x - 5) = -1.1917535925942099 5870(x - 5)$$

$$1.8917535925942099 5870x = 5.95876796297104$$

$$x = 3.149864753157222$$

and

$$y = 0.7x = 2.204905327210055669$$

So P is the point

$$(3.149864753157222, 2.204905327210055669).$$

5. *Susanne took a ride on a ferris wheel which had a radius of 22 meters. She began her ride at the lowest point on the wheel, just 4 meters above the ground. During her ride, her linear speed was 14 meters per second. How high above the ground was she 86 seconds into her ride?*

Because we are asked for her height, and because Susanne starts the ride at the lowest point, it doesn't matter which direction the wheel is rotating. We may as well think of it as moving counter-clockwise. Then Susanne's initial angular position is $-\frac{\pi}{2}$. Her angular speed is, from $v = r\omega$,

$$\omega = \frac{v}{r} = \frac{14}{22} = 0.636363636 \text{ radians/second}$$

so after 86 seconds, she has moved through an angle of

$$(86 \text{ sec})(0.636363636 \text{ radians/sec}) = 54.72727272 \text{ radians}$$

and hence her angular position is

$$\frac{-\pi}{2} + 54.72727272 = 53.1564764004778 \text{ radians.}$$

Thus her height above the center of the wheel is

$$r \sin(53.1564764004778) = 5.4556482852511009 \text{ meters.}$$

The center of the wheel is

$$4 + 22 = 26 \text{ meters}$$

above the ground, so Susanne is

$$26 + 5.4556482852511009 = 31.4556482852511009$$

meters above the ground 86 seconds into her ride.