Math 120 C and E - Autumn 2004
Mid-Term Exam Number One Solutions
October 21, 2004

1. *Let* $f(x) = x^2 + 2x, g(x) = x + 1.$

   (a) *Evaluate the expression*
   $$f(g(x)) - f(x)$$
   and simplify it as much as possible.

   $$f(g(x)) - f(x) = f(x+1) - f(x) = (x+1)^2 + 2(x+1) - (x^2 + 2x)$$
   $$= x^2 + 2x + 1 + 2x + 2 - x^2 - 2x = 2x + 3.$$

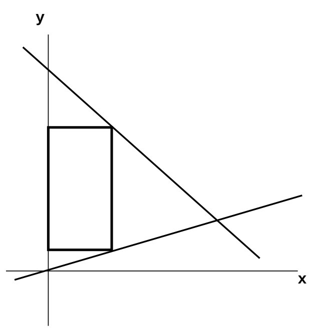
   (b) *The graphs of* $y = f(g(x))$ *and* $y = g(f(x))$ *are both parabolas. Which one has its vertex farther from the* x*-axis?*

   $$f(g(x)) = x^2 + 4x + 3 = (x+2)^2 - 1$$

   and
   $$g(f(x)) = x^2 + 2x + 1 = (x+1)^2$$

   so f(g(x)) has its vertex further from the x-axis.

2. *Consider the triangle formed by the* y*-axis and the lines* $y = \frac{1}{3}x$ *and* $y = 2 - x$*.*



Consider a rectangle placed inside this triangle, with one side on the y*-axis. An example of one such rectangle is shown in the figure. What is the area of the largest such rectangle?*

Let the width of such a rectangle be x. Then the height h is the difference of the upper line and the lower line:
$$h = (2-x) - \frac{1}{3}x = 2 - \frac{4}{3}x$$
The area of the rectangle is
$$A = xh = x(2 - \frac{4}{3}x) = -\frac{4}{3}x^2 + 2x = -\frac{4}{3}(x^2 - \frac{3}{2}x)$$
$$= -\frac{4}{3}((x - \frac{3}{4})^2 - \frac{9}{16}) = -\frac{4}{3}(x - \frac{3}{4})^2 + \frac{3}{4}.$$
so the maximum possible area is $\frac{3}{4}$.

3. *A plane begins its flight from a point 40 miles west and 30 miles south of Hampton airport. The radar at Hampton airport has a range of 40 miles (that is, it can detect anything within 40 miles). The plane flies in a straight line toward a point 80 miles north and 10 miles east of Hampton airport. The plane flies at a constant speed of 380 miles per hour. How long (in hours) will the plane be on the radar at Hampton airport?*

   If we impose a coordinate system with the origin at Hampton airport, then the limit of the radar's range is the circle

$$x^2 + y^2 = 40^2 = 1600$$

   and the path of the plane is the line

$$y = \frac{11}{5}(x + 40) - 30 = \frac{11}{5}x + 58 = 2.2x + 58$$

   We want the intersections points of the line and the circle:

$$x^2 + (2.2x + 58)^2 = 1600$$

$$5.84x^2 + 255.2x + 3364 = 1600$$

$$5.84x^2 + 255.2x + 1764 = 0$$

$$x = \frac{-255.2 \pm \sqrt{255.2^2 - 4(5.84)(1764)}}{2(5.84)} = -8.607798 \text{ or } -35.090832.$$

   The corresponding y-values are

$$y = 39.062844 \text{ and } y = -19.1998307$$

   The distance between these points is the distance of the plane's flight that it is on the radar:

$$\sqrt{(-8.607798 - (-35.090832))^2 + (39.062844 - (-19.1998307))^2} = 63.9991438 \text{ miles}$$

   Dividing this by the speed of the plane yields the time that the plane is on the radar:

$$\frac{63.9991438}{380} = 0.16841879 \text{ hours.}$$

4. *The population of Cambridge in 1990 was 12,000, and in 2000 it was 13,400.*

   Essex has a population of 18,100 in 1980, and in 1995 it was 17,500.

   Use linear models to describe the populations of Cambridge and Essex. According to your models, when will there be twice as many people in Cambridge as Essex?

   The linear model for Cambridge:

$$y = \frac{12000 - 13400}{1990 - 2000}(x - 1990) + 12000 = 140x - 266600.$$

   The linear model for Essex:

$$y = \frac{18100 - 17500}{1980 - 1995}(x - 1980) + 18100 = -40x + 97300.$$

If at time x there are twice as many people in Cambridge as in Essex, then

$$140x - 266600 = 2(-40x + 97300)$$

so

$$140x - 266600 = -80x + 194600$$
$$220x = 461200$$
$$x = 2096.36363636...$$

5. *Let*

$$g(x) = 2|x+1| + x^2 - 6x - 1.$$

(a) *Write the multipart rule for* $g(x)$*.*

$$g(x) = \begin{cases} 2(x+1) + x^2 - 6x - 1 & \text{if } x+1 \geq 0 \\ 2(-(x+1)) + x^2 - 6x - 1 & \text{if } x+1 < 0 \end{cases}$$

$$= \begin{cases} x^2 - 4x + 1 & \text{if } x \geq -1 \\ x^2 - 8x - 3 & \text{if } x < -1 \end{cases}$$

(b) *Find all solutions to the equation* $g(x) = 10$*.*

If $g(x) = 10$ then either

$$x^2 - 4x + 1 = 10$$

or

$$x^2 - 8x - 3 = 10$$

The first equation becomes

$$x^2 - 4x - 9 = 0$$

which gives solutions

$$x = \frac{4 \pm \sqrt{4^2 + 36}}{2} = 5.6055512 \text{ or } - 1.6055512.$$

Since $g(x) = x^2 - 4x + 1$ only if $x \geq -1$, only 5.6055512 is a solution of $g(x) = 10$.
The other equation becomes

$$x^2 - 8x - 13$$

which gives solutions

$$x = \frac{8 \pm \sqrt{8^2 + 4(13)}}{2} = -1.385164 \text{ or } 9.385164.$$

Since $g(x) = x^2 - 4x - 9$ only if $x < -1$, only -1.385164 is a solution to $g(x) = 10$.
Thus, the solutions to $g(x) = 10$ are 5.6055512 and -1.385164.