Solutions to Math 112 Spring 2014 Final Exam

1. (a) In the boxes under the given values, write 0, +, or - if the given value is zero, positive or negative, respectively. The first two are given as examples.

<table>
<thead>
<tr>
<th>$f(30)$</th>
<th>$f(69)$</th>
<th>$f'(5)$</th>
<th>$f''(8)$</th>
<th>$f'(45)$</th>
<th>$f''(55)$</th>
<th>$f'(10)$</th>
<th>$f''(80)$</th>
<th>$f'(62)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>0</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>0</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>

(b) Define $F(x) = \int_{0}^{x} f(t) dt$. The function $F(x)$ has a relative maximum when $x = 43$ or $x = 87$.

(c) Estimate for the following quantities from the graph.
$F(5) = 252.5$, $F'(25) = f(40) = 40$, $F''(55) = f'(55) = 0$

2. Let

$$g(x,y) = \frac{x^2 + 4xy + y^3}{2x + 3y}.$$

(a) Compute the partial derivatives $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

$$\frac{\partial g}{\partial x} = \frac{(2x + 4y)(2x + 3y) - (x^2 + 4xy + y^3)(2)}{(2x + 3y)^2}$$

$$\frac{\partial g}{\partial x} = \frac{(4x + 3y^2)(2x + 3y) - (x^2 + 4xy + y^3)(3)}{(2x + 3y)^2}$$

(b) ESTIMATE $\frac{g(3.2,5) - g(3,5)}{0.2}$ using partial derivatives.

$$g_x(3,5) = \frac{158}{441} \approx 0.358$$

(c) Compute the EXACT value of $\frac{g(3.2,5) - g(3,5)}{0.2}$.

$$\frac{199.24}{21.4} - \frac{194}{21} \approx 0.361$$
(d) What is the slope of the tangent line to the graph of \( g(2, y) \) at the point where \( y = 3 \)?

\[ g_y(2, 3) \approx 0.716 \]

3. Compute the following.

(a) \[ \frac{d}{dx} \left( e^{5x} (3x + 2)^7 \right) = 5e^{5x} (3x + 2)^7 + 21e^{5x} (3x + 2)^6 \]

(b) \[ \frac{d}{dx} \left( \ln \left( x^3 \sqrt{x + 5} \right) \right) = \frac{d}{dx} \left( 3 \ln x + \frac{1}{2} \ln(x + 5) \right) = \frac{3}{x} + \frac{1}{2(x + 5)} \]

(c) \[ \int \left( \frac{7t^2 - 1}{t^3} \right) dt = \int \left( \frac{7}{t} - t^{-3} \right) dt = 7 \ln t + \frac{1}{2t^2} + C \]

(d) \[ \int_0^{10} \left( \frac{4}{t + 1} + 3e^{t/2} \right) dt = 4 \ln(t + 1) + 6t^{1/2} \bigg|_0^{10} = 4 \ln 11 + 6e^5 - 6 \approx 894.07 \]

4. Suppose that an input of \( x \) thousand kilowatt hours of electricity and \( y \) hundred hours of labor in a factory result in the production of \( P(x, y) = -0.6x^2 + 0.2xy - 0.08y^2 + 24.6x - 2.2y \) thousand units of a product.

(a) This function is maximized at its critical point. What is the maximum number of units of product that can be produced in this factory?

\[ \frac{\partial P}{\partial x} = -1.2x + 0.2y + 24.6 = 0 \]

\[ \frac{\partial P}{\partial x} = 0.2x - 0.16y - 2.2 = 0 \]

gives \( x = 23 \) and \( y = 15 \). So the maximum value is \( P(23, 15) = 266.4 \) units.

(b) When 15 thousand kilowatt hours of electricity and 4 hundred hours of labor have been used, which one of the following would result in a higher increase in productivity? Explain your answer.

i. Increasing the energy use by 1 thousand kilowatt hours. The increase in productivity is approximately \( P_x(15, 4) = 7.4 \)
ii. Increasing the labor by 1 hundred hours. The increase in productivity is approximately

\[ P_y(15, 4) = 0.16 \]

So increasing the energy use would result in a higher increase in productivity.

5. The demand function for a certain product is \( p = 121 - 0.04x^2 \) and the supply function is \( p = 0.04x^2 + 0.8x + 73 \).

(a) Find the equilibrium point.

\[
121 - 0.04x^2 = 0.04x^2 + 0.08x + 73
\]

\[ 0.08x^2 + 0.8x - 48 = 0 \]

\[ x = \frac{-0.8 \pm \sqrt{0.64 + 0.32(48)}}{0.16} = -30 \text{ or } 20 \]

When \( x = 20 \), \( p = 121 - 0.04(20)^2 = 105 \). So the equilibrium point is \((20, 105)\).

(b) Find the consumer’s surplus.

\[
CS = \int_{0}^{20} (121 - 0.04x^2) \, dx - 20 \cdot 105 \approx 213.33
\]

6. In the picture below, the line is tangent to the parabola given by the equation \( y = -x^2 + 16x - 39 \) at the point where \( x = 6 \). Find the shaded area.

When \( x = 6 \), \( y = -36 + 16(6) - 29 = 21 \).

The derivative is \( y' = -2x + 16 \), so the slope of the tangent line is
\(-2(6) + 16 = 4.\)
The tangent line has equation
\[ y = 4x - 3. \]
The tangent line intersects the \(x\) axis when \(y = 4x - 3 = 0\) so \(x = 3/4.\)
The parabola intersects the \(x\) axis when \(-x^2 + 16 - 39 = 0\) so \(x = 3\) or \(x = 13.\)
The shaded area is the area of the triangle under the tangent line, which is \(21(6 - 3/4)/2 = 441/8\) minus the area under the parabola which is
\[ \int_3^6 (-x^2 + 16x - 39) \, dx = 36 \]
So the shaded area is \(153/8 = 19.125\)

7. The rate of ascent of two balloons A and B are given by the following graphs. The rate of ascent of Balloon A is \(a(t)\). The rate of ascent of Balloon B is \(b(t)\). At \(t = 0\), they are both 360 meters above the ground. The time is given in minutes and the rate of ascent is given in meters per minute.

(a) At \(t = 6\), Balloon A is GOING UP
(b) At \(t = 40\), Balloon B is GOING UP
(c) At \(t = 3\), Balloon A is SLOWING DOWN
(d) What are their lowest altitudes?
   Balloon A 360
   Balloon B 360-250=110
(e) What is the altitude of Balloon B at \(t = 40\) minutes?
\[ 360 - 250 + 250 + 20(50) = 1360 \]
(f) What is the maximum distance between them in the interval \(0 \leq t \leq 20\) and which balloon is closer to the ground when the distance between them is maximum?
The lines intersect when \(-50 + 5t = 40 - t\) so at \(t = 15.\) The distance between them is \(\frac{90 \cdot 15}{2} - 675\). Balloon B is closer to the ground.
8. The Marginal Cost and Total Revenue for producing Tops are given by the following functions

\[ MC(q) = 0.75q^2 - 13.5q + 73, \quad TR(q) = -2.5q^2 + 85q \]

where \( q \) is in thousands of units of Tops, \( TC \) is in thousands of dollars and \( MC \) is in dollars per unit of Tops.

(a) What is the cost of producing the 5001st unit of Tops?

\[ MC(5) = 24.25 \]

(b) When you sell the 14001st unit of Tops, does your profit increase or decrease? First, \( MR(x) = -5q + 85 \).

\[ MP(14) = MR(14) - MC(14) = -16 \]

so the profit decreases.

(c) At what quantity is the Profit maximized? When \( MC=MR \):

\[ 0.75q^2 - 13.5q + 73 = -5q + 85 \]

so

\[ q = \frac{8.5 \pm \sqrt{8.5^2 - 4(0.75)(-12)}}{1.5} = 12.60 \]

(d) Let \( a \) be your answer to part (c) above and let \( P \) be the profit function. Below are the graphs of \( TC \) and \( TR \). Label the graphs as \( TC \) and \( TR \). Show the quantities \( MC(a) \), \( MR(a) \) and \( P(a) \) on the graph.