NONRATIONAL COMPLETE INTERSECTIONS

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ABSTRACT. Kollár's method for proving nonrationality of hypersurfaces can be extended to more general complete intersections. A complete intersection of r very general hypersurfaces of degrees $d_1, \ldots, d_r$ in complex projective space $\mathbf{P}^N$ is not ruled, and therefore not rational, provided that $\sum d_i \geq 3N/4 + 2r + 1$.

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1. INTRODUCTION

An algebraic variety is rational if it contains a dense open subset that is isomorphic to a dense open subset of projective space.

By the classical adjunction formula, a smooth complete intersection of r hypersurfaces of degrees $d_1, \ldots, d_r$ in projective N-space cannot be rational if $\sum d_i \geq N + 1$. In [1] Kollár considered the $r = 1$ case, showing that a very general complex hypersurface is not ruled, and therefore not rational, already if its degree d satisfies

$$d \geq 2 \left\lceil \frac{N+2}{3} \right\rceil .$$

It is the object of this paper to show that Kollár's method may be extended to more general complete intersections. We prove that a very

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general complex complete intersection for which

$$\sum_{i=1}^{r} d_i \geq \frac{3}{4}N + 2r + 1,$$

where the notation is as above, is not ruled and therefore not rational.

Kollár's argument is based on Matsusaka's observation that, unlike rationality, ruledness specializes in families. He combines this observation with a degeneration of hypersurfaces proposed by Mori, reducing his nonrationality theorem to the problem of finding certain cyclic covers (of lower degree hypersurfaces in positive characteristic) that are not ruled. Kollár then shows it suffices to choose such a cyclic cover generically, relying for this on an algebraic version of the Morse Lemma and a certain criterion for nonruledness.

We summarize the main points of this paper as follows. One can study complete intersections using a simple variant of Mori's degeneration. The Morse Lemma must then be replaced with a singularity theory of morphisms to $\mathbf{A}^r$, where r denotes the codimension of the complete intersections. The structure of such morphisms becomes increasingly complicated as r grows, but one does not have to control them too much, as the criterion for nonruledness used by Kollár extends to singular varieties.

For simplity of proof we degenerate to characteristic 3 instead of 2. It is for this reason that our lower bound has a $\frac{3}{4}$ instead of Kollár's better $\frac{2}{3}$.

In this paper we keep the singularity theory at a minimum and develop what we need from scratch for lack of a suitable reference covering positive characteristics. We expect that with a developed theory one would be able to extend Totaro's result [4] that very general complex hypersurfaces of degree $d \geq 2\lceil (N+1)/3 \rceil$ are not stably rational.

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2. Notation

We denote the ring of integers by $\mathbf{Z}$ and the field of rational numbers by $\mathbf{Q}$.

Let $(R, \mathfrak{m}, k)$ be a local ring. A subring $k_0 \subseteq R$ is called a *coefficient field* if the natural map $k_0 \to R/\mathfrak{m}$ is an isomorphism. Let $f \in R[[t]]$ be a power series. Denote by $f(0)$ the image of f under the R-algebra map $R[[t]] \to R$ that sends $t \mapsto 0$. Denote by $\bar{f}$ the image of f under the map $R[[t]] \to k[[t]]$ induced by the quotient map $R \to R/\mathfrak{m}$. Denote by $f' \in R[[t]]$ the formal derivative of f with respect to t. Complete local rings are assumed Noetherian.

Suppose a base field k is fixed. An *algebraic scheme* is a separated k-scheme of finite type. A *variety* is an integral algebraic scheme. Let Z be a variety.

We say that Z is *rational* if there exists a birational map $\mathbf{P}^n \dashrightarrow Z$. We say that Z is *ruled* if there exists a birational map $Y \times \mathbf{P}^1 \dashrightarrow Z$, where Y is a variety. We say that Z is *(separably) uniruled* if there exists a (separable) dominant generically finite map $Y \times \mathbf{P}^1 \dashrightarrow Z$, where Y is a variety. Note: rational $\Rightarrow$ ruled $\Rightarrow$ separably uniruled $\Rightarrow$ uniruled.

We denote the *function field* of Z, that is, the stalk of $\mathscr{O}_Z$ at the generic point, by $k(Z)$. Let F be an $\mathscr{O}_Z$-module. Define its *torsion subsheaf* $\operatorname{Tors} F$ to be the kernel of the natural map $F \to F \otimes_{\mathscr{O}_Z} k(Z)$.

Suppose that k is the field of complex numbers. If all points outside a countable union of proper subvarieties of Z have a given property, we say that that property holds for a *very general* point of Z.

Let X be a scheme. Let E be a locally free $\mathscr{O}_X$-module. We denote the *vector bundle* associated to E by $\pi : \mathbf{V}(E) \to X$. Thus

$$\mathbf{V}(E) = \operatorname{Spec}_X \operatorname{Sym}_{\mathscr{O}_X}(E^\vee).$$

Recall this carries a *universal* (sometimes called *tautological*) section $h \in \Gamma(\mathbf{V}(E), \pi^*E)$ which corresponds to Id_E under the identification

$$\mathscr{H}om(E, E) = E \otimes E^\vee \subseteq E \otimes \operatorname{Sym} E^\vee = \pi_*\pi^*E.$$

Thus, for every scheme T over X, pullback of h defines a bijection

$$\operatorname{Hom}_X(T, \mathbf{V}(E)) \xrightarrow{\sim} \Gamma(T, E_T).$$

Let $f : X \to S$ be a morphism of schemes. We denote by $\Omega^i_{X/S}$, or simply Ω^i_X if S is understood, the ith exterior power of the sheaf of Kaehler differentials $\Omega_{X/S}$. Thus we allow for torsion in Ω^i_X.

Let M be an $\mathscr{O}_X$-module. We denote the corresponding ith *sheaf of principal parts* by $\mathscr{P}^i_{X/S}(M)$, or $\mathscr{P}^i_X(M)$ if S is understood. This is an $\mathscr{O}_X$-module equipped with a sheaf morphism

$$d^i : M \to \mathscr{P}^i_X(M),$$

the universal $f^{-1}\mathscr{O}_S$-linear differential operator of order i on M. Recall that the image of d^i generates $\mathscr{P}^i_X(M)$ as an $\mathscr{O}_X$-module. If $S = \operatorname{Spec} K$ is the spectrum of a field and $x \in X(K)$ is a rational point, then d^i induces an isomorphism

$$M/\mathfrak{m}_x^{i+1}M \xrightarrow{\sim} \mathscr{P}^i_X(M) \otimes_{\mathscr{O}_X} k(x).$$

We write $\mathscr{P}^i_{X/S}$ or $\mathscr{P}^i_X$ instead of $\mathscr{P}^i_{X/S}(\mathscr{O}_X)$.

Suppose that X is smooth over S. We say that $x_1, \ldots, x_n \in \Gamma(X, \mathscr{O}_X)$ are *étale coordinates* on X if one of the following three equivalent conditions hold: (1) their differentials generate $\Omega_{X/S}$; (2) their differentials form a basis for $\Omega_{X/S}$; and (3) the map $x = (x_1, \ldots, x_n) : X \to \mathbf{A}^n_S$ is étale.

Let $x_1, \ldots, x_n$ be étale coordinates on X. Let $h \in \mathscr{O}_X$ be a section. We define its *partial derivatives* $\partial_i h \in \mathscr{O}_X$ via the equation $dh = \sum_{i=1}^n \partial_i h \cdot dx_i$. We define the *Hessian* of h to be the matrix $H(f) := [\partial_i \partial_j h] \in \mathscr{O}_X^{\oplus n \times n}$.

3. Main result: degeneration

Recall the following construction. Let X be a scheme. Let L an invertible sheaf. Let $s \in \Gamma(X, L^{\otimes p})$, where $p > 0$. Let $\pi : \mathbf{V}(L) \to X$ be the line bundle associated with L. Let $y \in \Gamma(\mathbf{V}(L), \pi^* L)$ denote its universal section. Let

$$X[s^{1/p}] := \{y^{\otimes p} = s\} \subseteq \mathbf{V}(L).$$

The restriction $\pi : X[s^{1/p}] \to X$ is called the *cyclic cover* of X determined by s.

For $i = 1, \ldots, r$, let L_i be an invertible sheaf on X and let $s_i \in \Gamma(X, L_i^{\otimes p_i})$, where $p_i > 0$. Then

$$X[s_1^{1/p_1}, \ldots, s_r^{1/p_r}] := X[s_1^{1/p_1}] \times_X \cdots \times_X X[s_r^{1/p_r}] \to X$$

is called the *abelian cover* (or $\oplus_{i=1}^r \mathbf{Z}/(p_i)$*-cover*) of X determined by $s_1, \ldots, s_r$.

Proposition 3.1. *Let k be an infinite field of characteristic $p \geq 3$. Let X be an integral smooth proper scheme of dimension n over k. Let $r \leq \frac{1}{2}n - 1$ be an integer. For each $i = 1, \ldots, r$, let L_i be an invertible sheaf, let $V_i \subseteq \Gamma(X, L_i^{\otimes p})$ be a finite dimensional k-linear subspace such that*

$$V_i \otimes_k \mathscr{O}_X \xrightarrow{d^2 \otimes 1} \mathscr{P}^2_X(L_i^{\otimes p}) \tag{1}$$

is surjective for $i = 1, \ldots, r$. Suppose that

$$\omega_X \otimes L_1^{\otimes p} \otimes \cdots \otimes L_r^{\otimes p}$$

is big. For $i = 1, \ldots, r$, let $s_i \in V_i$ be a general section. Then $X[s_1^{1/p}, \ldots, s_r^{1/p}]$ is (geometrically integral and) not separably uniruled.

The condition that (1) be surjective is equivalent to the following one: For every geometric point $x \in X(\bar{k})$, the natural map

$$V_i \otimes_k \bar{k} \to L_i^{\otimes p}/\mathfrak{m}_x^3 L_i^{\otimes p}$$

is surjective.

Proof. This follows from Proposition 4.1 and Lemma 5.6. The latter applies by Corollary 8.3 and Proposition 8.5. □

Consider the following variant of Mori's dengeration of a hypersurface [3, Example 4.3]. Let R be a discrete valuation ring. Let K denote the fraction field of R, let k be its residue field and $t \in R$ a uniformizer. Let P denote the weighted projective space

$$\mathbf{P}(1^{N+1}, a_1, \ldots, a_r) = \operatorname{Proj} R[x_0, \ldots, x_N, y_1, \ldots, y_r],$$

where on the polynomial ring on the right the variables $x_0, \ldots, x_N, y_1, \ldots, y_r$ are assigned respective degrees $1, \ldots, 1, a_1, \ldots, a_r$. Let $p > 0$ be an integer. For $i = 1, \ldots, r$, let $f_i, g_i \in R[x_0, \ldots, x_N]$ be homogeneous of respective degrees pa_i and a_i. Let $Z \subseteq P$ be defined by the equations $y_i^p - f_i = ty_i - g_i = 0$ with $i = 1, \ldots, r$. Then the generic fiber of $Z \to \operatorname{Spec} R$ is the subscheme

$$\{g_1^p - t^p f_1 = \cdots = g_r^p - t^p f_r = 0\} \subseteq \mathbf{P}_K^N,$$

whereas the special fiber is the $(\mathbf{Z}/(p))^r$-cover of

$$\{g_1 = \cdots = g_r = 0\} \subseteq \mathbf{P}_k^N$$

determined by the $f_i \in \mathscr{O}(a_i)^{\otimes p}$ with $i = 1, \ldots, r$.

The main result of this paper is the following.

Theorem 3.2. *Let $N, r, d_1, \ldots, d_r$ be positive integers. Suppose that $d_1, \ldots, d_r$ have a common prime factor $p \geq 3$. Suppose furthermore that these integers satisfy*

$$r \leq \frac{N-2}{3} \qquad \text{and} \qquad \sum_{i=1}^{r} d_i > \frac{p}{p+1}(N+1).$$

Then the complete intersection of r very general hypersurfaces of degrees $d_1, \ldots, d_r$ in complex projective N-space is not ruled.

Proof. Let s be an indeterminate. Let $\mathfrak{p}$ denote the prime ideal $p\mathbf{Z}[s] \subset \mathbf{Z}[s]$. Thus $\mathfrak{p}$ corresponds to the generic point of the fiber over p of the natural map

$$\operatorname{Spec} \mathbf{Z}[s] \to \operatorname{Spec} \mathbf{Z}.$$

Let $R = \mathbf{Z}[s]_{\mathfrak{p}}$. Then R is a discrete valuation ring whose residue field $k = (\mathbf{Z}/(p))(s)$ is infinite of characteristic p and whose fraction field K is contained in $\mathbf{Q}(s)$ and hence embeds in the complex numbers.

Let $\bar{g}_1, \ldots, \bar{g}_r \in k[x_0, \ldots, x_N]$ be homogeneous polynomials of degrees $d_1/p, \ldots, d_r/p$ such that

$$X := \{\bar{g}_1 = \cdots = \bar{g}_r = 0\} \subset \mathbf{P}_k^N$$

is a smooth complete intersection.

For $i = 1, \ldots, r$, let $L_i = \mathscr{O}_X(d_i/p)$ and let $V_i \subseteq \Gamma(X, L_i^{\otimes p})$ be the image of $\Gamma(\mathbf{P}_k^N, \mathscr{O}(d_i))$. Let $\bar{k}$ be an algebraic closure of k. Because $d_i \geq 2$, the map $V_i \otimes_k \bar{k} \to (\mathscr{O}_X/\mathfrak{m}_x^3)(d_i)$ is surjective for any rational point $x \in X(\bar{k})$. The invertible sheaf

$$\omega_X \otimes L_1^{\otimes p} \otimes \cdots \otimes L_r^{\otimes p} = \mathscr{O}_X(-N - 1 + (\tfrac{1}{p} + 1) \textstyle\sum d_i)$$

is big because by assumption $\sum d_i > (N+1)/(1+\frac{1}{p})$. Let $n = N - r$; this is the dimension of X. Note that

$$r \leq \frac{(N-2)}{3} \Leftrightarrow r \leq \frac{n}{2} - 1.$$

Hence it follows from Proposition 3.1 that there exist homogeneous polynomials $\bar{f}_1, \ldots, \bar{f}_r \in k[x_0, \ldots, x_N]$ of degrees $d_1, \ldots, d_r$ such that $X[\bar{f}_1^{1/p}, \ldots, \bar{f}_r^{1/p}]$ is geometrically integral and not separably uniruled, hence not ruled.

Let $\pi : Z \to \operatorname{Spec} R$ be the variant of Mori's degeneration described above constructed with any homogeneous lifts

$$g_1, \ldots, g_r, f_1, \ldots, f_r \in R[x_0, \ldots, x_N]$$

of the $\bar{g}_i$ and $\bar{f}_i$. Then π is equidimensional and flat with geometrically integral fibers. Applying the Kollár-Matsusaka result [2, Theorem 1.8.3], we conclude that the generic fiber of π is not ruled. Thus there exists a complex complete intersection in $\mathbf{P}^N$ of multidegree $(d_1, \ldots, d_r)$ that is not ruled. Another application of [2, Theorem 1.8.3] to the family of all complete intersections yields the result. $\square$

Corollary 3.3. *Let $N, r, d_1, \ldots, d_r$ be positive integers. Let $p \geq 3$ be a prime number. For $i = 1, \ldots, r$, let $e_i \in \{0, 1, \ldots, p-1\}$ be the remainder of the division of d_i by p. Suppose that*

$$r \leq \frac{N-2}{3} \qquad \text{and} \qquad \sum_{i=1}^{r} (d_i - e_i) > \frac{p}{p+1}(N+1).$$

Then the complete intersection of r very general hypersurfaces of degrees $d_1, \ldots, d_r$ in complex projective N-space is not ruled.

Proof. Degenerate the equations of degree that aren't evenly divisible by p to equations that are times generic linear forms and apply Matsusaka's result [2, Theorem IV.1.6.2]. $\square$

Note that the Corollary applies as soon as

$$\sum_{i=1}^{r} d_i \geq \frac{p}{p+1} N + 1 + r(p-1) \tag{2}$$

and $r \leq (N-2)/3$. When (2) holds, the condition on r becomes superfluous, since when it fails the conclusion of the Corollary holds trivially by the adjuntion formula. Setting $p = 3$ yields the result in the introduction.

4. A necessary condition for separable uniruledness

The proof of the main theorem in this paper requires the following strengthening of a result of Kollár's [2, Lemma V.5.1].

Lemma 4.1. *Let X be an integral, generically smooth, proper scheme over a field k. Let M be a big line bundle on X. Let $i > 0$ be an integer. Let T denote the torsion subsheaf of Ω^i_X. Suppose that there exists an injection of $\mathscr{O}_X$-modules $M \hookrightarrow \Omega^i_X/T$. Then X is not separably uniruled.*

Hence we are not assuming X smooth. Note that if resolutions of singularities are known to exist, then the lemma stated follows from the one in [2]. Our proof is modelled in that of Kollár's. Because X is no longer assumed smooth, it replaces the use of the deformation theory of morphisms $\mathbf{P}^1 \to X$ by an elementary argument.

Proof. Aiming for a contradiction, suppose that X is separably uniruled. Then there exists an integral variety Y together with a generically finite, dominant and separable rational map $f : Y \times \mathbf{P}^1 \dashrightarrow X$.

The restriction

$$f : \operatorname{Spec} k(Y) \times \mathbf{P}^1 = \mathbf{P}^1_{k(Y)} \dashrightarrow X$$

is a morphism by properness of X. Thus by shrinking Y we may assume f is a morphism.

Being f separable and generically finite, the natural map

$$f^*\Omega^1_X \to \Omega^1_{Y \times \mathbf{P}^1} \tag{3}$$

is surjective at the generic point of $Y \times \mathbf{P}^1$. Because X is generically smooth, we have

$$\dim_{k(X)}(\Omega_X \otimes k(X)) = \dim X = \dim Y + 1 \leq \dim_{k(Y)}(\Omega_{Y \times \mathbf{P}^1} \otimes k(Y)).$$

It follows that (3) is injective at the generic point of $Y \times \mathbf{P}^1$ and that Y is generically smooth. Shrinking Y, we may assume Y is smooth.

The ith exterior power of (3) vanishes on f^*T by smoothness of Y. The resulting composition

$$f^*M \to f^*(\Omega^i_X/T) \to \Omega^i_{Y\times\mathbf{P}^1}$$

is injective at the generic point of $Y\times\mathbf{P}^1$. Because f^*M is torsion free, the composition is in fact injective everywhere. More generally, for any integer $m>0$, the map

$$f^*M^{\otimes m} \to (\Omega^i_{Y\times\mathbf{P}^1})^{\otimes m}$$

is injective at the generic point of $Y\times\mathbf{P}^1$ and therefore everywhere.

Let $V\subseteq Y\times\mathbf{P}^1$ be the open subset of consisting of the the points $z\in Y\times\mathbf{P}^1$ such that the map

$$f^*M\otimes k(z)\to\Omega^i_{Y\times\mathbf{P}^1}\otimes k(z)$$

is injective. Thus, given a section $s\in f^*M^{\otimes m}$ and $z\in V$, to check whether s vanishes at z it suffices to check whether the image of s in $(\Omega^i_{Y\times\mathbf{P}^1})^{\otimes m}$ vanishes at z. Note that V contains the generic point of $Y\times\mathbf{P}^1$, so is dense.

By Chow's lemma, there exists a birational morphism $\rho: X'\to X$ with X' projective over k. The pullback ρ^*M is again big, so by projectivity there exist Cartier divisors A and E on X', the first ample and the second effective, and an integer $m>0$ such that $\rho^*M^{\otimes m}=\mathscr{O}_{X'}(A+E)$. Let $U\subseteq X$ be a dense open subset over which ρ is an isomorphism and such that $\rho^{-1}(U)$ is disjoint from E.

Because f is generically finite, there exists a dense open subset $U'\subseteq X$ over which f is finite.

Let $W=f^{-1}(U\cap U')\cap V$. This is a dense open subset of $Y\times\mathbf{P}^1$. Let $\bar{k}$ be an algebraic closure of k. Then there exists a geometric point $y\in Y(\bar{k})$ such that the intersection $y\times\mathbf{P}^1\cap W$ is nonempty. This intersection in fact contains infinitely many $\bar{k}$-points, so by definition of U' there are two of them, $z_1,z_2\in(y\times\mathbf{P}^1)\cap W(\bar{k})$, which have distinct images in $U'\cap U\subseteq X$. By definition of U, these two points z_1 and z_2 are separated by a global section of $f^*M^{\otimes m}$ for some $m>0$ sufficiently large and divisible. By definition of V, it follows that $z_1,z_2\in y\times\mathbf{P}^1$ are separated by global sections of $(\Omega^i_{Y\times\mathbf{P}^1})^{\otimes m}$. This is impossible, since

$$H^0(Y\times\mathbf{P}^1,(\Omega^i_{Y\times\mathbf{P}^1})^{\otimes m})=H^0(Y,(\Omega^i_Y)^{\otimes m})\otimes_k H^0(\mathbf{P}^1,\mathscr{O}_{\mathbf{P}^1}). \tag{4}$$

Let us verify (4). We compute

$$\begin{aligned}(\wedge^i\Omega_{Y\times\mathbf{P}^1})^{\otimes m} &= (\wedge^i(\mathrm{pr}_1^*\Omega_Y\otimes\mathrm{pr}_2^*\Omega_{\mathbf{P}^1}))^{\otimes m}\\ &= (\wedge^i\mathrm{pr}_1^*\Omega_Y\oplus(\wedge^{i-1}\mathrm{pr}_1^*\Omega_Y\otimes\mathrm{pr}_2^*\Omega^1_{\mathbf{P}^1}))^{\otimes m}\\ &= \oplus_{a+b=m}\mathrm{pr}_1^*((\wedge^{i-1}\Omega_Y)^{\otimes a}\otimes(\wedge^i\Omega_Y)^{\otimes b})\otimes\mathrm{pr}_2^*\Omega_{\mathbf{P}^1}^{\otimes a}.\end{aligned}$$

The Künneth formula now gives (4), since $\Omega_{\mathbf{P}^1}^{\otimes a}$ has no global sections for $a > 0$. $\square$

5. Ramified covers

Let X be a scheme over a field k of characteristic $p > 0$. Let $L_1, \ldots, L_r$ be invertible sheaves on X. For $i = 1, \ldots, r$, let $s_i \in \Gamma(X, L_i^{\otimes p})$ be a section. Let

$$Y := X[s_1^{1/p}, \ldots, s_r^{1/p}].$$

For $i = 1, \ldots, r$, let $y_i \in \Gamma(\mathbf{V}(L_i), L_i)$ denote the tautological section. Then Y is the closed subscheme of

$$U := \mathbf{V}(L_1) \times_X \cdots \times_X \mathbf{V}(L_r) = \mathbf{V}(\oplus_{i=1}^r L_i)$$

defined by the vanishing of the sections

$$y_i^{\otimes p} - s_i \in \Gamma(U, L_i^{\otimes p_i})$$

with $i = 1, \ldots, r$. Let π denote both the projection $U \to X$ and its restriction to Y.

Lemma 5.1. *The morphism $\pi : Y \to X$ is finite, flat and induces a homeomorphism of underlying spaces.*

Proof. Note that $\pi : Y \to X$ factors as

$$Y = X[s_1^{1/p_1}, \ldots, s_r^{1/p_r}] \to \cdots \to X[s_1^{1/p_1}, s_2^{1/p_2}] \to X[s_1^{1/p_1}] \to X.$$

Each step is a cyclic cover. If A is a ring and $f \in A$, then $A[y]/(y^p - f)$ is free A-module of rank p. It follows that $\pi : Y \to X$ is finite and flat. If K is a field of characteristic p and $f \in K$, then the spectrum of $K[y]/(y^p - f)$ consists of a single point. Thus the fibers of $\pi : Y \to X$ consist of single points. The result follows. $\square$

There is a natural connection

$$d : L_i^{\otimes p} \to \Omega_X^1 \otimes L_i^{\otimes p}$$

defined as follows. If τ is a local generator of L_i and $t = f\tau^p$ is a local section of $L_i^{\otimes p}$, set

$$dt := df \otimes \tau^p.$$

This is independent of the choices made and defines d.

Suppose that X is connected and smooth over k. Let

$$S_1 := \{ds_1 \wedge \cdots \wedge ds_r = 0\} \subseteq X.$$

Thus S_1 is defined by the vanishing of a section of the locally free sheaf $\Omega_X^r \otimes (\otimes_{i=1}^r L_i^{\otimes p})$. Suppose that S_1 has codimension at least 2 in X.

Lemma 5.2. *Y is locally a complete intersection over k. The inverse image $\pi^{-1}S_1$ is the locus where Y is not smooth. It follows that Y is (geometrically) integral and normal.*

Proof. That Y is locally a complete intersection in the smooth k-scheme U is clear. This implies that it is Cohen-Macaulay. The second statement is also clear, since $\pi^{-1}S_1 \subseteq Y$ is the locus where the differentials of the equations defining Y in U are not independent. It follows from Serre's criterion that the connected components of Y are integral and normal. As X is connected, so is Y. $\square$

Write $E = \oplus_{i=1}^r L_i^{\otimes p}$. The direct sum of the natural connections on $L_i^{\otimes p}$ yields a connection

$$d : E \to \Omega_X^1 \otimes E.$$

Let $s = (s_1, \ldots, s_r) \in \Gamma(X, E)$. View ds as a map $E^\vee \to \Omega_X^1$. Let $C = \operatorname{coker}(ds)$ and $Q = (\wedge^{n-r} C)^{\vee\vee}$.

Lemma 5.3. *Q is invertible and isomorphic to*

$$\omega_X \otimes L_1^{\otimes p} \otimes \cdots \otimes L_r^{\otimes p}. \tag{5}$$

Proof. Note that C is locally free and

$$0 \to E^\vee \to \Omega_X^1 \to C \to 0 \tag{6}$$

is exact over $X \setminus S_1$. Thus Q is a reflexive coherent sheaf of rank 1 on a regular scheme, hence invertible. The sequence (6) yields an isomorphism of Q with (5) over $X \setminus S_1$. Because S_1 has codimension at least 2, this isomorphism extends uniquely to one over all of X. $\square$

Lemma 5.4. *Let $x \in X$ be a k-rational point. Choose trivializations $L_i \cong \mathscr{O}_X$ near x. For $i = 1, \ldots, r$, let $f_i \in \mathscr{O}_{X,x}$ be the section corresponding to $s_i \in L$. Suppose that are formal coordinates $x_1, \ldots, x_n$ around $x \in X$ with the property that $f_i = f_i(0) + x_i$ for $i = 1, \ldots, r$. For every $i \in \{r, \ldots, n\}$ such that $\partial_i f_r \neq 0$, define*

$$\eta_i := \frac{dx_r \cdots \widehat{dx_i} \cdots dx_r}{\partial_i f_r} \in \Omega_X^{n-r} \otimes \operatorname{Frac} \widehat{\mathscr{O}}_{X,x}$$

Then the image of η_i in $Q \otimes \operatorname{Frac} \widehat{\mathscr{O}}_{X,x}$ lies inside, and is a generator of, $Q \otimes \widehat{\mathscr{O}}_{X,x}$.

Proof. By definition

$$C = \Omega_X / \langle df_1, \ldots, df_r \rangle = \Omega_X / \langle dx_1, \ldots, dx_{r-1}, df_r \rangle.$$

Hence $\wedge^{n-r} C$ is generated by the products $dx_r \cdots \widehat{dx_i} \cdots dx_n$ with $r \leq i \leq n$. Because

$$0 = df_r \wedge (dx_r \cdots \widehat{dx_i} \cdots \widehat{dx_j} \cdots dx_n),$$

these satisfy relations

$$\partial_j f_r \cdot dx_r \cdots \widehat{dx_i} \cdots dx_r = \pm \partial_i f_r \cdot dx_r \cdots \widehat{dx_j} \cdots dx_r$$

for all $r \leq i, j \leq n$. Thus $\eta_i = \pm \eta_j$ for all $i, j \in \{r, \ldots, n\}$ for which they are defined. Hence they are regular and generate Q over $X \setminus S_1$. Because S_1 has codimension at least 2, it follows that they are regular and generate Q over all of X. □

Lemma 5.5. *The natural map $\pi^*\Omega_X \to \Omega_Y$ factors through a unique map $\pi^* C \to \Omega_Y$. Taking wedge powers and double duals yields an injection*

$$\pi^* Q \hookrightarrow \wedge^{n-r} \Omega_Y \otimes k(Y).$$

Proof. Let $I_Y \subseteq \mathscr{O}_U$ be the ideal sheaf defining Y. There is a natural surjection

$$\pi^* E^\vee \twoheadrightarrow I_Y$$

that sends a section σ of $\pi^* E^\vee$ to its pairing with the section of $\pi^* E$ defining Y in U. Thus there is an exact sequence

$$\pi^* E^\vee \xrightarrow{d_Y} \Omega^1_U|_Y \to \Omega^1_Y \to 0.$$

Over an open subset of X where the L_i are trivial, $\pi^* E^\vee$ is trivial too and the morphism d_Y is given by the differentials

$$p y_i^{p-1} dy_i - ds_i$$

with $i = 1, \ldots, r$.

Because k has characteristic p, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\Omega_X & \longrightarrow & \Omega_U|_Y & \longrightarrow & \Omega_{U/X}|_Y & \longrightarrow & 0 \\ & & {\scriptstyle -ds}\uparrow & & {\scriptstyle d_Y}\uparrow & & \uparrow & & \\ 0 & \longrightarrow & \pi^* E^\vee & = & \pi^* E^\vee & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

is commutative. In it the upper row is exact because U is locally an affine space over X. The Snake Lemma hence yields the exact sequence

$$0 \to \pi^* C \to \Omega_Y \to \Omega_{U/X}|_Y \to 0.$$

Taking duals of coherent sheaves commutes with flat pullback. There is thus a map $\pi^* Q \to (\wedge^{n-r} \Omega_Y)^{\vee\vee}$, which is injective at the generic point of Y, hence everywhere. □

Suppose that $p \geq 3$ and $r \leq \frac{n}{2} - 1$. Suppose that S_1 is smooth of dimension $r-1$ over k. Let $\rho : B \to Y$ denote the blowup of the scheme theoretic inverse image $\pi^{-1}S_1 \subseteq Y$. Let T denote the torsion subsheaf $\operatorname{Tors}\Omega_B^{n-r}$.

Lemma 5.6. *The composition*

$$\rho^*\pi^*Q \to \rho^*\Omega_Y^{n-r} \otimes k(B) \to \Omega_B^{n-r} \otimes k(B)$$

factors through an inclusion $\rho^*\pi^*Q \hookrightarrow \Omega_B^{n-r}/T$.

Proof. The factoring morphism

$$\rho^*\pi^*Q \dashrightarrow \Omega_B^{n-r}/T \hookrightarrow \Omega_B^{n-r} \otimes k(B)$$

is unique if it exists. By descent, it suffices to check it does on a fpqc cover of B. Thus we may assume k is algebraically closed. It suffices to show the factoring morphism exists after pulling back to the spectrum of $\mathscr{O}_{B,b}$ for all closed points $b \in B$. In fact, by descent, we may work on completed local rings of B.

Let $b \in B$ be a closed point. Write $y = \rho(b)$ and $x = \pi(y)$. The existence of the factoring morphism is clear over points of $X \setminus S_1$, so assume $x \in S_1$.

Shrinking X to a Zariski open neighborhood of x, we may fix trivializations $L_i \cong \mathscr{O}_X$ for all i. Let $f_i \in \Gamma(X, \mathscr{O}_X)$ correspond to $s_i \in \Gamma(X, L_i^{\otimes p})$ under this isomorphism.

Quite generally, if Z is a scheme, $\alpha : F \to F'$ is a map of locally free sheaves and $z \in S_{j+1}(\alpha)$ is a point, then the ideal of $S_j(\alpha)$ at z is contained in $\mathfrak{m}_z^2$. See Section 7. Thus the smoothness of the first critical locus $S_1 = S_1(f_1, \ldots, f_r)$ implies that second $S_2(f_1, \ldots, f_r)$ is empty. By Lemma 7.1 we may find an isomorphism $\widehat{\mathscr{O}}_{X,x} \cong k[[x_1, \ldots, x_n]]$ such that $f_i - f_i(0) = x_i$ for $i = 1, \ldots, r$ and

$$f = f_r - f_r(0) = g + x_{2r-1}^2 + \cdots + x_n^2$$

with $g \in k[[x_1, \ldots, x_{2r-2}]]$.

Identify U with the affine r-space $\mathbf{A}^r \times X$ over X with coordinates $y_1, \ldots, y_r$. Effecting the coordinate change $y_i \mapsto y_i - (f_i(0))^{1/p}$ for $i = 1, \ldots, r$ yields the isomorphism

$$\widehat{\mathscr{O}}_{Y,y} \cong \frac{k[[x_1, \ldots, x_n, y_1, \ldots, y_r]]}{\langle y_1^p - x_1, \ldots, y_{r-1}^p - x_{r-1}, y_r^p - f \rangle}.$$

Let B' denote the base change of $B \to Y$ to $\operatorname{Spec} \widehat{\mathscr{O}}_{Y,y}$. Because S_1 is smooth, the generators $\partial_r f, \ldots, \partial_n f$ of its ideal form a regular sequence.

The same is true for their pullbacks along the flat map $\pi : Y \to X$. Therefore

$$B' = \operatorname{Proj} \frac{\widehat{\mathscr{O}}_{Y,y}[T_r, \dots, T_n]}{\langle T_i \partial_j f - T_j \partial_i f \rangle_{r \le i,j \le n}}.$$

To complete the proof we now check that the rational differential form $\eta_i \in \Omega_X^{n-r} \otimes \operatorname{Frac} \widehat{\mathscr{O}}_{X,x}$ of Lemma 5.4 pulls back to a (regular) section of Ω_B^{n-r}/T on

$$D_+(T_i) = \operatorname{Spec} \frac{\widehat{\mathscr{O}}_{Y,y}[T_r, \dots, \widehat{T_i}, \dots, T_n]}{\langle \partial_j f - T_j \partial_i f \rangle_{r \le j \le n; j \ne i}},$$

for all $i = r, \dots, n$.

Suppose that $i \le 2r-2$. We observe that $\partial_j f = 2x_j$ for all $j \ge 2r-1$ and compute:

$$\begin{aligned}
\eta_i =& \frac{dx_r \cdots \widehat{dx_i} \cdots dx_n}{\partial_i f} \\
=& dx_r \cdots \widehat{dx_i} \cdots dx_{2r-2} \frac{d(\frac{1}{2} T_{2r-1} \partial_i f) \cdots d(\frac{1}{2} T_n \partial_i f)}{\partial_i f} \\
=& \frac{1}{2^{n-2r+1}} dx_r \cdots \widehat{dx_i} \cdots dx_{2r-2} \Bigg((\partial_i f)^{n-2r} dT_{2r-1} \cdots dT_n \\
& + (\partial_i f)^{n-2r-1} \sum_{j=2r-1}^{n} (-1)^{j-2r-1} T_j d(\partial_i f) dT_{2r-1} \cdots \widehat{dT_j} \cdots dT_n \Bigg).
\end{aligned}$$

Thus η_i is regular along $D_+(T_i)$ if $n \ge 2r+1$.

The same argument shows that, if $n \ge 2r+2$, then η_i is regular on any $D_+(T_i)$ with $i > 2r-2$. □

6. Morse lemma

We record the following lemma for future reference.

Lemma 6.1. *Let $(R, \mathfrak{m})$ be a complete local ring. Let $f \in R[[t]]$ be a power series. Suppose that $\bar{f}(0) = 0$ and $\bar{f}'(0) \ne 0$. Then there exists a unique factorization $f = u(t-a)$ with $u \in R[[t]]^\times$ a unit and $a \in \mathfrak{m}$.*

Lemma 6.1 will be applied in the case that $R = k[[x_1, \dots, x_n]]$; it then reduces to an instance of the implicit function theorem. As stated, Lemma 6.1 a special case of the Weierstrass preparation theorem.

Lemma 6.2. *Let $(R, \mathfrak{m}, k)$ be a complete local ring. Suppose that R contains a coefficient field. Suppose that* $\operatorname{char}(k) \ne 2$. *Let $f \in R[[x]]$. Suppose that*

$$\bar{f}'(0) = 0 \quad \textit{and} \quad \bar{f}''(0) \ne 0.$$

Then there exist $f_0 \in R$, $h \in (\mathfrak{m}, x)$ *and* $u \in k^\times$ *such that*

$$f = f_0 + uh^2$$

and the R*-algebra map* $R[[y]] \to R[[x]]$ *that sends* $y \mapsto h$ *is an isomorphism.*

Proof. By Lemma 6.1, there exists $a \in \mathfrak{m}$ such that $f' \in (x - a)$. Let $f_0 := f(a) \in R$. Let $k = f - f_0 \in R[[x]]$. Note that $k(a) = 0 \in R$, so we may write $k = (x - a)k_1$ for some $k_1 \in R[[x]]$. Differentiating, we obtain

$$k' = k_1 + (x - a)k_1'.$$

It follows from $k' = f' \in (x-a)$ that $k_1 \in (x-a)$. Hence $k = (x-a)^2 k_2$ for some $k_2 \in R[[x]]$. Note that $2\bar{k}_2(0) = \bar{f}''(0) \neq 0$. By Hensel's lemma applied to the complete local ring $R[[x]]$, the power series $k_2(0)k_2$, a square modulo $(\mathfrak{m}, x)$, is also a square in $R[[x]]$. Setting $u = 1/k_2(0)$ and $h = (x - a)\sqrt{k_2(0)k_2}$, we obtain $k = uh^2$. □

Lemma 6.3 (Morse)**.** *Let* $(R, \mathfrak{m}, k)$ *be a complete local ring. Suppose that* R *contains a coefficient field. Suppose that* $\operatorname{char}(k) \neq 2$. *Let* $f \in R[[x_1, \dots, x_n]]$. *Suppose that*

$$\frac{\partial \bar{f}}{\partial x_1}(0) = \cdots = \frac{\partial \bar{f}}{\partial x_n}(0) = 0 \quad \text{and} \quad \det\left(\frac{\partial^2 \bar{f}}{\partial x_i \partial x_j}\right)(0) \neq 0.$$

Then there exist $f_0 \in R$, $h_1, \dots, h_n \in (\mathfrak{m}, x_1, \dots, x_n)$ *and* $u_1, \dots, u_n \in k^\times$ *such that*

$$f = f_0 + u_1 h_1^2 + \cdots + u_n h_n^2$$

and the R*-algebra map* $R[[y_1, \dots, y_n]] \to R[[x_1, \dots, x_n]]$ *that sends* $y_i \mapsto h_i$ *is an isomorphism.*

Proof. Applying a k-linear automorphism of $R[[x_1, \dots, x_n]]$, we may assume the Hessian matrix of $\bar{f}$ is diagonal. Let $R_{n-1} = R[[x_1, \dots, x_{n_1}]]$. Viewing f as an element of $R_{n-1}[[x_n]]$, we may write $f = f_{n-1} + u_n h_n^2$ as in Lemma 6.2. Then $f_{n-1} \in R_{n-1}$ satisfies the same hypotheses as f and the result follows by descending induction on n. □

7. Critical loci

Let X be a scheme. Let E and F be locally free sheaves of respective ranks e and f on X. Let $m = \min(e, f)$. Let $\alpha : E \to F$ be an $\mathscr{O}_X$-linear map. For each integer $p \geq 0$, define the *pth degeneracy locus* of α to be the subscheme

$$S_p(\alpha) := \{\wedge^{m-p+1}\alpha = 0\} \subseteq X.$$

Thus the support of $S_p(\alpha)$ is the set of points $x \in X$ such that $\alpha \otimes k(x) : E(x) \to F(x)$ has rank at most $m-p$ and there are natural inclusions

$$\emptyset = S_{m+1}(\alpha) \subseteq \cdots \subseteq S_1(\alpha) \subseteq S_0(\alpha) = X.$$

Let S be a scheme. Suppose that X is smooth over S. Let Y be a second smooth S-scheme and let $f : X \to Y$ be a morphism. Define its *pth critical locus* $S_p(f)$ of f to be the pth degeneracy locus of the differential $df : T_{X/S} \to f^*T_{Y/S}$.

We note in passing that these degeneracy loci coincide with the Fitting subschemes of the coherent sheaf $\Omega_{X/Y}$, which may be defined much more generally (for example, as soon as Y is locally Noetherian and $f : X \to Y$ is of finite type).

Define the *target corank* of f at a point $x \in X$ to be the integer $r - \operatorname{rank} df_x$.

Target corank 1 points lie in $S_p(f) \setminus S_{p+1}(f)$, where p satisfies $m-p = r-1$ and $m = \min(n, r)$. This implies that either $n = r-1$ and $p = 0$, or $r \leq n$ and $p = 1$. In the former case the locus of target corank 1 points is $X \setminus S_1(f)$. In the latter it is a subscheme of $X \setminus S_2(f)$. Let us find equations defining this subscheme.

Choosing étale coordinates on Y, we may assume $Y = \mathbf{A}^r_S$, so that $f = (f_1, \ldots, f_r)$ with $f_i \in \Gamma(X, \mathscr{O}_X)$. Let $z \in X \setminus S_2(f)$. Then $r-1$ among $df_1, \ldots, df_r$ are linearly independent in $\Omega_{X/S} \otimes k(z)$. Without loss of generality, these are the first $r-1$. Hence, by shrinking X to a neighborhood of z, we may find étale coodinates $x_1, \ldots, x_n \in \Gamma(X, \mathscr{O}_X)$ with $x_i = f_i$ for $i = 1, \ldots, r-1$. Then

$$\begin{aligned} S_1(f) &= \{dx_1 \wedge \cdots \wedge dx_{r-1} \wedge df_r = 0\} \\ &= \left\{ \frac{\partial f_r}{\partial x_r} = \cdots = \frac{\partial f_r}{\partial x_n} = 0 \right\}. \end{aligned}$$

Let $U \subseteq S_1(f)$ be the locus where the last $n-r+1$ columns of the Hessian $H(f_r)$ are linearly independent. Then U coincides with the set where the fibers of $S_1(f) \to S$ are smooth of dimension $r-1$. If S is regular, then $U \to S$ is smooth by miracle flatness.

Lemma 7.1. *Let k be a field. Suppose* $\operatorname{char}(k) \neq 2$. *Let X be a smooth scheme over k. Let $f = (f_1, \ldots, f_r) : X \to \mathbf{A}^r$ a morphism. Suppose $1 \leq r \leq n$. Let $x \in X$ be a k-rational point at which $S_1(f) \setminus S_2(f)$ is smooth of dimension $r-1$. Then there exists a renumbering of the f_i and an isomorphism of local k-algebras*

$$\widehat{\mathscr{O}}_{X,x} \cong k[[x_1, \ldots, x_n]]$$

under which $f_i = x_i + c_i$ *with* $c_i \in k$ *for* $i = 1, \dots, r-1$ *and*

$$f_r = g + u_{2r-1}x_{2r-1}^2 + \dots + u_n x_n^2$$

with $g \in k[[x_1, \dots, x_{2r-2}]]$ *and* $u_{2r-1}, \dots, u_n \in k^\times$.

Proof. Observe the following. Let V be a vector space of dimension n. Let $E \subseteq V$ be a vector subspace of codimension c. Let $q : V^\vee \to E^\vee$ be induced by the inclusion. Let $h : V \to V^\vee$ be a linear map. Then

$$\operatorname{rank}(q \circ h|_E) \geq \operatorname{rank}(h|_E) - c. \tag{7}$$

Because $x \notin S_2(f)$, after a renumbering of the f_i there are formal coordinates $x_1, \dots, x_n$ around x such that $f_i = x_i + c_i$ with $c_i \in k$ for $i = 1, \dots, r-1$. Furthermore, because $S_1(f)$ is smooth at x, the last $n-(r-1)$ columns of the Hessian matrix $Hf_r(0)$ are independent. It follows from (7) that the minor of size $n-(r-1)$ in the bottom right of $Hf_r(0)$ has rank at least $n-2(r-1)$. Effecting a k-linear change of the last $n-r+1$ coordinates, we may assume that this minor is a diagonal matrix

$$\begin{bmatrix} d_r & & & \\ & d_{r+1} & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

with $d_{2r-1}, \dots, d_n$ nonzero. Now apply Lemma 6.3 to f_r with

$$R = k[[x_1, \dots, x_{2r-2}]].$$

□

8. General position

The following result is well known.

Lemma 8.1 (Serre)**.** *Let X be a scheme of finite type over a field k. Let E be a locally free sheaf of rank e on X. Let $s_0, s_1, \dots, s_N \in \Gamma(X, E)$. Suppose $s_1, \dots, s_N$ generate E. Let $S = \operatorname{Spec} k[t_1, \dots, t_N]$ and let*

$$s_t = s_0 + t_1 s_1 + \dots + t_N s_N \in \Gamma(X_S, E_S).$$

Note that s_t may be viewed as a map $X_S \to \mathbf{V}(E)$. Let $V \subseteq \mathbf{V}(E)$ be a locally closed subscheme of pure dimension d. Then there exists a dense open subset $U \subseteq S$ over which $s_t^{-1}(V)$ is flat of pure relative dimension $d-e$. If X and V are smooth and k has characteristic zero, then U may be chosen so that $s_t^{-1}(V)$ is smooth over U.

The conclusion of this result should be interpreted as allowing for the possibility that $s_t^{-1}(V)_U$ is empty.

Proof. Observe the morphism $s_t : X \times \mathbf{A}^N \to \mathbf{V}(E)$ is surjective and smooth of relative dimension $N - e$. (This question is local on X, so it suffices to consider the easy case in which E is trivial, generated by $s_1, \ldots, s_e$.) Thus $s_t^{-1}(V)$ has pure dimension $d + N - e$ and is smooth if X and V are. Nonempty open sets $U \subseteq S$ over which $s_t^{-1}(V)$ is flat (resp. smooth, under the hypotheses on smoothenss and characteristic) exist by well known generic flatness (resp. generic smoothness) theorems. The relative dimension of $s_t^{-1}(V)_U \to U$ follows from flatness and the pure dimensions of its source and target. ☐

The following proposition is well known.

Proposition 8.2. *Let X be a scheme. Let E and F be locally free sheaves on X. Let $H = \mathbf{V}(\mathscr{H}om(E, F))$ and denote by $\pi : H \to X$ the projection. Let $h \in \Gamma(H, \pi^*\mathscr{H}om(E, F))$ be the universal section. Then $S_p(h) \setminus S_{p+1}(h) \subset H$ is smooth of relative dimension $ef - p(|e-f|+p)$ over X.*

Corollary 8.3. *Let X be a smooth scheme of dimension n a field k. For $i = 1, \ldots, r$, let $f_{i1}, \ldots, f_{iN_i} \in \Gamma(X, \mathscr{O}_X)$ be such that $df_{i1}, \ldots, df_{iN_i}$ generate $\Omega_{X/k}$ as an $\mathscr{O}_X$-module. Let*

$$S = \operatorname{Spec} k[\{t_{ij} : 1 \le i \le r; 1 \le j \le N_i\}].$$

For each i, let $f_i = t_{i1}f_{i1} + \cdots + t_{iN_i}f_{iN_i} \in \Gamma(X_S, \mathscr{O}_{X_S})$. Let $f = (f_1, \ldots, f_r) : X_S \to \mathbf{A}^r_S$. Then there exists a dense open subset $U \subseteq S$ over which $S_p(f) \setminus S_{p+1}(f)$ is flat of relative dimension $n - p(|n - r| + p)$. If k has characteristic zero, then U may be chosen so that $S_p(f) \setminus S_{p+1}(f)$ is smooth over U.

Proof. Let $y_1, \ldots, y_r$ be the coordinates on $\mathbf{A}^r_S$. For $i = 1, \ldots, r$ and $j = 1, \ldots, N_i$, identify $df_{ij} \in \Gamma(X_S, \Omega_{X/S})$ with the composition

$$T_{X/S} \xrightarrow{df_{ij}} \mathscr{O}_X \xrightarrow{\partial/\partial y_i} T_{\mathbf{A}^r_S/S}.$$

Then the df_{ij} generate $E := \mathscr{H}om(T_{X/S}, T_{\mathbf{A}^r_S/S})$. The result now follows from Lemma 8.1 applied to E and $V = S_p(h) \setminus S_{p+1}(h)$ as in Proposition 8.2. ☐

The following lemma is well known.

Lemma 8.4. *Let S be a scheme. Let X be a smooth scheme over S. Fix étale coordinates $x_1, \ldots, x_n \in \Gamma(X, \mathscr{O}_X)$ on X. Let*

$$d + H : \mathscr{O}_X \to \Omega_{X/S} \oplus \mathscr{O}_X^{\oplus n \times n}$$

be the sheaf morphism that sends a section of $\mathscr{O}_X$ to the pair consisting of its differential and Hessian. Let A denote the sheaf of symmetric

n by n matrices with entries in $\mathscr{O}_X$ and whose diagonal entries are multiples of 2. Then the $\mathscr{O}_X$-module generated by the image of $d + H$ is $\Omega_{X/S} \oplus A$.

Proposition 8.5. *Let X be a smooth scheme of dimension n a field k. Suppose that $\operatorname{char}(k) \neq 2$. Let $r \leq n$. For $i = 1, \ldots, r$, let $f_{i1}, \ldots, f_{iN_i} \in \Gamma(X, \mathscr{O}_X)$ be such that $d^2 f_{i1}, \ldots, d^2 f_{iN_i}$ generate $\mathscr{P}^2_{X/k}$ as an $\mathscr{O}_X$-module. Let*

$$S = \operatorname{Spec} k[\{t_{ij} : 1 \leq i \leq r; 1 \leq j \leq N_i\}].$$

For each i, let $f_i = t_{i1} f_{i1} + \cdots + t_{iN_i} f_{iN_i} \in \Gamma(X_S, \mathscr{O}_{X_S})$. Let $f = (f_1, \ldots, f_r) : X_S \to \mathbf{A}^r_S$. Then there exists a dense open subset $U \subseteq S$ over which $S_1(f) \setminus S_2(f)$ is smooth of relative dimension $r - 1$.

Proof. It suffices to show that the generic fiber of $S_1(f)/S_2(f) \to S$ is smooth. Let L denote the function field of S.

For $l = 1, \ldots, r$, let $U_l \subseteq X_L$ be the open subset where

$$df_1, \ldots, \widehat{df_l}, \ldots, df_r$$

are linearly independent in $\Omega_{X_L/L}$. Then $X_L \setminus S_2(f)$ is the union of the U_l, so it suffices to prove that $S_1(f) \cap U_l$ is smooth for $l = 1, \ldots, r$.

Fix l. Without loss of generality, $l = r$. Let $k \subset K$ be the field extension generated by the t_{ij} with $i \leq r-1$, so that $L = K(t_{r1}, \ldots, t_{rN_r})$. Then $f_1, \ldots, f_{r-1}$ are identified with global sections of $\mathscr{O}_{X_K}$. Let $X' \subseteq X_K$ be the open subset where $df_1, \ldots, df_{r-1}$ are independent in $\Omega_{X_K/K}$.

The question of smoothness of $S_1(f) \cap U_r$ is Zariski local on $U_r = X'_L$, so we may assume there exist étale coordinates $x_1, \ldots, x_n \in \Gamma(X', \mathscr{O}_{X'})$ on X' such that $x_i = f_i$ for $i = 1, \ldots, r-1$.

Let Q denote the locally free sheaf $\Omega_{X'/K}/\langle dx_1, \ldots, dx_{r-1}\rangle$. Let $Z \subset \mathbf{V}(Q)$ denote the zero section.

Let A denote the sheaf of symmetric n by n matrices with entries in $\mathscr{O}_{X'}$. Let $h \in \Gamma(\mathbf{V}(A), A)$ denote its universal section. Let $D \subseteq \mathbf{V}(A)$ be the subscheme defined by the determinants of the maximal minors of the submatrix formed by the last $n - r + 1$ columns of h.

Note that f_r defines morphisms

$$df_r : X'_L \to \mathbf{V}(Q) \qquad \text{and} \qquad Hf_r : X'_L \to \mathbf{V}(A)$$

where Hf_r denotes the Hessian of f_r. By Section 7,

$$S_1(f) \cap X'_L = df_r^{-1}(Z),$$

the singular locus of which is $df_r^{-1}(Z) \cap Hf_r^{-1}(D)$.

By Lemma 8.4, the sections

$$df_{r1} + Hf_{r1}, \ldots, df_{rN_r} + Hf_{rN_r}$$

generate $Q \oplus A$. By Lemma 8.6, $Z \times_L D$ has codimension

$$(n - r + 1) + r = n + 1$$

in $\mathbf{V}(Q \oplus A)$. It follows from Lemma 8.1 that

$$df_r^{-1}(Z) \cap Hf_r^{-1}(D) = (df_r, Hf_r)^{-1}(Z \times_L D)$$

has codimension $n + 1$ in X'_L and is therefore empty. □

Lemma 8.6. *Let X be a scheme. Let $n \geq 1$ be an integer. Let $N = n(n+1)/2$. Let $\mathbf{A}^N_X$ have coordinates $\{x_{ij}\}_{1 \leq i \leq j \leq n}$. Let M be the symmetric matrix*

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix}.$$

Let $1 \leq q \leq n$. Let $D \subseteq \mathbf{A}^N_X$ be the subscheme defined by the determinants of the maximal minors of the submatrix formed by the last q columns of M. Then the fibers of $D \to X$ have dimension $N - n + q - 1$.

Proof. We argue by induction on q. If $q = 1$, we have a column vector M of n indeterminates. Its vanishing clearly has relative codimension $n = n - q + 1$ over R. If $q > 1$, consider the submatrix M' obtained by deleting from M its first row and column. By induction, the vanishing in $\mathbf{A}^N_X$ of the determinants of the maximal minors of M' has relative codimension

$$(n - 1) - (q - 1) + 1 = n - q + 1$$

over X. Hence to prove the lemma it suffices to study the codimension of D on the open subsets $\{\det m \neq 0\} \subset \mathbf{A}^N_X$, where m is a maximal minor of M'. Fix one such maximal minor m and restrict attention to the open subset of $\mathbf{A}^N_X$ where its determinant is invertible.

Let $I \subseteq \{1, \ldots, n\}$ be the set of rows intersecting m. Thus I has cardinality $q - 1$ and $1 \notin I$. For each $i \notin I$, let m_i denote the minor of M formed by the rows of M that intersect m and the ith row. Then the ideal of $D \subseteq \mathbf{A}^N_X$ is generated by the $\det m_i$ with $i \notin I$. Expanding $\det m_i$ along the first column, we obtain

$$\det m_i = \pm x_{i1} \det m + R_i,$$

where R_i is an integral polynomial in the x_{ij} which does not involve x_{11} and, moreover, does not involve any x_{j1} with $j \notin I$ provided that $i \geq 2$. The result follows. □

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