BRANCHED COVERS OF CONES AND KODAIRA DIMENSION

LUCAS BRAUNE

Abstract. In this note I compute for branched covers of cones (1) the Iitaka dimensions of their canonical divisors and (2) their Kodaira dimensions. These calculations show that among such covers there are examples in all dimensions of varieties with at most isolated quotient singularities for which the quantities (1) and (2) are distinct.

Contents

1. Introduction

The Kodaira dimension is a birational invariant of smooth projective varieties. We define the Kodaira dimension of a singular projective variety to be that of any desingularization.

The value of the Kodaira dimension of a (possibly singular) projective variety is of consequence to its geometry. For example, if the Kodaira dimension is nonnegative, then the variety cannot be covered by rational curves.

Computing this invariant for a singular variety can be difficult when a resolution of singularities is not readily available. It such circumstances, it may still be possible to compute the Iitaka dimension of the canonical bundle of the singular variety.

Date: December 24, 2016.

Thus it is of interest to know when, for a given (say, normal and $\mathbb{Q}$-Gorenstein) projective variety Y with resolution of singularities Z, the equation

$$\kappa(\omega_Y) = \kappa(\omega_Z), \tag{1}$$

between the Iitaka dimensions of their canonical bundles holds.

It is a standard fact that (1) holds if the singularities of Y are at worst canonical. It holds for some varieties with slightly worse singularities as well, notably for the moduli spaces of stable curves $\overline{M}_g$ [2, Theorem 1] and for all toric varieties [1, Theorem 8.2.3]. As we shall see below, (1) holds for cones, provided that the singularity at the vertex is not very bad (e.g. provided it is log canonical).

In this note I record calculations on branched covers of cones which provide examples of varieties with reasonably mild singularities for which (1) does *not* hold. Among these examples are the following.

Proposition. *Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety. Let $C \subseteq \mathbb{P}^{n+1}$ be (the normalization of) the projective cone over X. Assume the singularity at the vertex of C is $\mathbb{Q}$-Gorenstein and not canonical.*

Let $f : Y \to C$ be a degree m cyclic cover branched along degree md hypersurface section of C that is disjoint from the vertex of this cone. Assume that either $d = 1$ and $m = 3$, or $d = m = 2$.

Then

$$\kappa(\omega_Y) = \dim X + 1 \quad \textit{while} \quad \kappa(\omega_Z) = \dim X.$$

This proposition applies for example when X is the Veronese variety of degree D and dimension N, provided that D is large in the sense that $D > N + 1$.

For the general result of the computations of $\kappa(\omega_Y)$ and $\kappa(\omega_Z)$, see sections 15 and 16. When reading those, refer to section 13 for notation.

We carry out the basic calculations that follow over an arbitrary base field.

2. Projective cones

Consider the rational map

$$\pi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$$
$$(x_0, \ldots, x_{n+1}) \mapsto (x_0, \ldots, x_n).$$

This map may be described as the projection from the point $P = (0, \ldots, 0, 1)$ onto the hyperplane $\{x_{n+1} = 0\} \cong \mathbb{P}^n$ inside $\mathbb{P}^{n+1}$.

In terms of invertible sheaves, it is the rational map to $\mathbb{P}^n$ defined on $\mathbb{P}^{n+1}$ by the n sections

$$x_0, \ldots, x_n \in \mathscr{O}_{\mathbb{P}^{n+1}}(1)$$

or, equivalently, by the map of sheaves

$$\oplus_{i=0}^n \mathscr{O}_{\mathbb{P}^n} \cdot x_i \to \mathscr{O}_{\mathbb{P}^{n+1}}(1).$$

Let $X \subseteq \mathbb{P}^n$ be a closed subscheme. Let $C \subseteq \mathbb{P}^{n+1}$ be the (scheme-theoretic) closure of the inverse image of X via π in $\mathbb{P}^{n+1} \setminus \{P\}$.

Thus generators for the homogeneous ideal of X in $\mathbb{P}^n$, when viewed as elements of $k[x_0, \ldots, x_{n+1}]$, where k denotes the base field, furnish generators for the homogeneous ideal of C in $\mathbb{P}^{n+1}$.

We refer to C as the *projective cone* over X and to P as the *vertex* of this projective cone.

3. Blow ups of projective cones

Restricting $\pi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ we obtain rational map $C \dashrightarrow X$, which we still denote by π. To resolve $\pi : C \dashrightarrow X \subseteq \mathbb{P}^n$, we blow up its base locus.

Namely, we are given the map of sheaves

$$\oplus_{i=0}^{n} \mathscr{O}_C \cdot x_i \to \mathscr{O}_C(1).$$

Tensoring with $\mathscr{O}_C(-1)$, we obtain

$$\oplus_{i=0}^{n} \mathscr{O}_C(-1) \cdot x_i \twoheadrightarrow J \subseteq \mathscr{O}_C. \tag{2}$$

It is clear that the subscheme of the cone C defined ideal sheaf J is the vertex P viewed as a reduced closed subscheme of C.

Let $p : \mathrm{Bl}_P C \to C$ be the blowing up of J. Denote $\mathrm{Bl}_P C$ by B. Pulling back (2) to B, we obtain the folowing diagram, in which E denotes the exceptional divisor of $p : B \to C$.

$$\begin{array}{ccccc} \oplus_{i=0}^{n} p^* \mathscr{O}_C(-1) \cdot x_i & \twoheadrightarrow & p^* J & \longrightarrow & \mathscr{O}_B \\ & & \downarrow & \nearrow & \\ & & \mathscr{O}_B(-E) & & \end{array}$$

Tensoring with $p^* \mathscr{O}_C(1)$, we obtain a surjection

$$\oplus_{i=0}^{n} \mathscr{O}_B \cdot x_i \twoheadrightarrow p^* \mathscr{O}_C(1)(-E).$$

This surjection defines a morphism $q : B \to \mathbb{P}^n$. This morphism factors through X and fits a commutative diagram of dominant rational maps:

$$\begin{array}{ccc} B & & \\ {\scriptstyle p}\downarrow & \searrow{\scriptstyle q} & \\ C & \overset{\pi}{\dashrightarrow} & X \end{array}$$

Thus blowing up the vertex of C resolves π as claimed.

4. The projective bundle structure of B

The natural inclusion

$$q^* \mathscr{O}_X(1) = (p^* \mathscr{O}_C(1))(-E) \hookrightarrow p^* \mathscr{O}_C(1)$$

and the section $x_{n+1} \in p^* \mathscr{O}_C(1)$ together define a surjection

$$q^*(\mathscr{O}_X(1) \oplus \mathscr{O}_X) \twoheadrightarrow (p^* \mathscr{O}_C)(1). \tag{3}$$

In the remainder of this section we show that this line bundle quotient gives $q : B \to X$ the structure of the projective line bundle $\mathbb{P}(\mathscr{O}_X(1) \oplus \mathscr{O}_X)$ over X.

Let S be a scheme with a map $f : S \to X \subseteq \mathbb{P}^n$ and a surjection

$$f^*(\mathscr{O}_X(1) \oplus \mathscr{O}_X) \twoheadrightarrow L,$$

where L is an invertible sheaf on S. Precomposing this map with the pull back along f of the surjection

$$\oplus_{i=0}^{n} \mathscr{O}_X \cdot x_i \twoheadrightarrow \mathscr{O}_X(1),$$

we obtain a surjection

$$(\oplus_{i=0}^{n} \mathscr{O}_S) \oplus \mathscr{O}_S \twoheadrightarrow L.$$

Let $g : S \to \mathbb{P}^{n+1}$ be the corresponding morphism.

We claim that $\pi g = f$ over $g^{-1}(\mathbb{P}^{n+1} \setminus \{P\})$. Indeed, πg is defined by the morphism of sheaves

$$\oplus_{i=0}^{n} \mathscr{O}_S \cdot x_i \to L.$$

This is surjective on $g^{-1}(\mathbb{P}^{n+1} \setminus \{P\})$ and therefore the composition

$$f^*\mathscr{O}_X(1) \to f^*(\mathscr{O}_X(1) \oplus \mathscr{O}_X) \twoheadrightarrow L$$

is an isomorphism there. This proves our claim.

The inverse image $g^{-1}\{P\}$ is defined by the inverse image ideal sheaf $g^{-1}J$, which is the image of

$$\oplus_{i=0}^{n} L^{-1} \cdot x_i \to \mathscr{O}_S$$

and hence equal to $f^*\mathscr{O}_X(1) \otimes L^{-1}$. This is an invertible sheaf, so by the universal property of the blowing up $p : B \to C$, there exists a unique morphism $h : S \to B$ such that $g = ph$.

To see that the morphism h is the one required by the universal property of $\mathbb{P}(\mathscr{O}_X(1) \oplus \mathscr{O}_X)$, note that the pullback along h of (3) coincides the given map $f^*(\mathscr{O}_X(1) \oplus \mathscr{O}_X) \twoheadrightarrow L$. Indeed,

$$h^*(p^*\mathscr{O}_C(1)) = g^*\mathscr{O}_C(1) = L$$

and

$$h^*\mathscr{O}_B(-E) = f^*\mathscr{O}_X(1) \otimes L^{-1},$$

so

$$h^*((p^*\mathscr{O}_C(1))(-E)) = f^*\mathscr{O}_X(1).$$

5. A cover of B by affine bundles

The projective line bundle $q : B \to X$ is covered by the two open subsets $B \setminus H_{n+1}$ and $B \setminus E$, where H_{n+1} denotes the divisor $\{x_{n+1} = 0\} \subseteq B$.

There is a natural morphism of $\mathscr{O}_X$-algebras

$$\operatorname{Sym} \mathscr{O}_X(1) \to q_*\mathscr{O}_{B \setminus H_{n+1}} \tag{4}$$

given by $x_i \mapsto x_i/x_{n+1}$ for $i = 0, \dots, n$. Similarly, there is a natural morphism of $\mathscr{O}_X$-algebras

$$\operatorname{Sym} \mathscr{O}_X(-1) \to q_*\mathscr{O}_{B \setminus E} \tag{5}$$

which over $\{x_i \neq 0\} \subseteq X$ is given by $1/x_i \mapsto x_{n+1}/x_i$.

Using x_i to trivialize $\mathscr{O}_X(1)$ over $\{x_i \neq 0\} \subseteq X$, we see that these two morphisms of $\mathscr{O}_X$-algebras are isomorphisms and give $B \setminus H_{n+1}$ and $B \setminus E$ the structure of the total spaces of respectively $\mathscr{O}_X(-1)$ and $\mathscr{O}_X(1)$. Thus the decomposition

$$B = (B \setminus H_{n+1}) \cup (B \setminus E)$$

restricts to the standard decomposition of $\{x_i \neq 0\} \times \mathbb{P}^1_{x_i, x_{n+1}}$ into affine open subsets.

6. Poles along H_{n+1} and E

Suppose X is smooth, so that B is as well.

Let L be a line bundle on X. For later use, we show in this section that, given $m \in \mathbb{Z}$,

$$H^0(B, q^*L(mH_{n+1})) = \oplus_{i=0}^m H^0(X, L \otimes \mathscr{O}_X(i)) \tag{6}$$

$$H^0(B, q^*L(mE)) = \oplus_{i=0}^m H^0(X, L \otimes \mathscr{O}_X(-i)). \tag{7}$$

From the preceding section, we have

$$\begin{aligned} q_*((q^*L)|_{B\setminus H_{n+1}}) &= L \otimes \operatorname{Sym} \mathscr{O}_X(1) \\ &= L \otimes (\oplus_{i=0}^\infty \mathscr{O}_X(i)). \end{aligned}$$

The equality (6) will follow once we show that a section in the ith summand on the right has a pole of order exactly i along H_{n+1}.

Indeed, the case $m < 0$ of (6) follows from the case $m = 0$ coupled with the observation that $H_{n+1} \subset B$ is a section of $q : B \to X$. Namely, if the pullback of a section of L on X vanishes along H_{n+1}, then the section was already zero. Thus it suffices to consider poles, that is, $m \geq 0$.

The equality (7) follows from (6) with L replaced by $L \otimes \mathscr{O}_X(-m)$. It may also be proved in exactly the same way.

Now let us prove (6).

Let $l \geq 0$. Let $U \subseteq X$ be an open subset. Let $s \in L \otimes \mathscr{O}_X(l)(U)$ be a nonzero section. Let $x \in U$ be a point at which s does not vanish.

We have a surjection

$$L \otimes_k k[x_0, \dots, x_n]_l = L \otimes_{\mathscr{O}_X} \operatorname{Sym}^l(\oplus_{i=0}^n \mathscr{O}_X \cdot x_i) \twoheadrightarrow L \otimes_{\mathscr{O}_X} \mathscr{O}_X(l).$$

It follows that, after shrinking U around x if necessary, there is a section $s \in L(U)$ and a homogeneous polynomial P of degree l such that

$$t \otimes P(x_1, \dots, x_n) = s.$$

Shrinking U further, we may assume that t and $P(x_1, \dots, x_n)$ generate over this open subset L and $\mathscr{O}_X(l)$, respectively. The inverse image $q^{-1}U$ intersects H_{n+1} on a dense open subset, since q is a projective line bundle. On this subset, t is a generator of q^*L and $P(x_1, \dots, x_n)$ is a generator of $q^*\mathscr{O}_X(1)$.

Viewed as a section of $p^*\mathscr{O}_C(l)$ via the identification

$$q^*\mathscr{O}_X(1) = p^*\mathscr{O}_C(1)(-E),$$

the section $P(x_1, \dots, x_n)$ has no zeros or poles along H_{n+1}, which is disjoint from E. Hence the rational function

$$P(x_1, \dots, x_n)/x_{n+1}^l \in \mathscr{O}_B(q^{-1}U \setminus H_{n+1})$$

has a pole of order exactly l along H_{n+1}.

Since s viewed as a rational section of q^*L equals this rational function times the generator t, it follows s has a pole of order exactly l on H_{n+1}, as claimed.

7. Euler sequence for Grassmannians

In this section we give a description of the differentials on projective bundles and, more generally, on relative Grassmannians.

Let X be a scheme and F a locally free sheaf of rank r. Let $G = \operatorname{Gr}_n(F)$ be the Grassmannian parametrizing rank n quotients of F. Let Q be the universal quotient of F on G, and let $A \subseteq F \otimes \mathscr{O}_G$ be the corresponding universal subbundle. Thus we have an exact sequence:

$$0 \to A \to F \otimes_{\mathscr{O}_X} \mathscr{O}_G \to Q \to 0$$

For $i = 1, 2$, let $\pi_i : G \times_X G \to G$ denote the projection. The diagonal of $G \times_X G$ is precisely the subscheme where the composition

$$\pi_1^* A \hookrightarrow F \otimes_{\mathscr{O}_X} \mathscr{O}_{G \times G} \twoheadrightarrow \pi_2^* Q$$

is zero. Thus if $\Phi \in \pi_1^* A^\vee \otimes \pi_2^* Q$ denotes the composition above, the ideal defining $\Delta(G) \subseteq G \times_X G$ is the image I of the multiplication map

$$\pi_1^* A \otimes \pi_2^* Q^\vee \xrightarrow{\cdot \Phi} \mathscr{O}_{G \times_X G}.$$

Pulling this surjection onto I back along the diagonal $\Delta : G \to G \times_X G$, we obtain

$$A \otimes Q^\vee \twoheadrightarrow \Omega^1_{G/X},$$

which is in fact an isomorphism since the two sheaves are locally free of the same rank. Taking the first short exact sequnce displayed in this section and tensoring it with $Q^\vee$, we obtain the *Euler sequence* of the Grassmannian:

$$0 \to \Omega^1_{G/X} \to F \otimes Q^\vee \to Q \otimes Q^\vee \to 0.$$

8. Differentials on the blow up of a projective cone

Return to the setup of section 4. Assume X is smooth.

Recall that $B \cong \mathbb{P}(\mathscr{O}_X(1) \oplus \mathscr{O}_X)$ and denote $F := \mathscr{O}_X(1) \oplus \mathscr{O}_X$. Then the Euler sequence for this projective bundle reads

$$0 \to \Omega_{\mathbb{P}F/X} \to q^* F \otimes \mathscr{O}_{\mathbb{P}F}(-1) \to \mathscr{O}_{\mathbb{P}F} \to 0.$$

The sequence of sheaves of differentials associated with the morphism $q : B \to X$ is

$$0 \to q^* \Omega_X \to \Omega_{\mathbb{P}F} \to \Omega_{\mathbb{P}F/X} \to 0.$$

This is exact on the left because $q : B \to X$ is locally a product.

Putting these two together, we obtain

$$\begin{aligned} \omega_{\mathbb{P}F} &= q^* \omega_X \otimes \det \Omega_{\mathbb{P}F/X} \\ &= q^*(\omega_X \otimes \det F) \otimes \mathscr{O}_{\mathbb{P}F}(-\operatorname{rank} F). \end{aligned}$$

In terms of p and q, this is expressed

$$\begin{aligned} \omega_B &= q^*(\omega_X \otimes \mathscr{O}_X(1)) \otimes p^* \mathscr{O}_C(-2) \\ &= q^*(\omega_X \otimes \mathscr{O}_X(-1))(-2E). \end{aligned}$$

9. Cone singularities

Assume the cone C is normal. This is equivalent to X being projectively normal as defined in an exercise in Hartshorne's book [3].

Note that this condition is satisfied when the singularity at the vertex P of C is finite quotient, as is the case when X is a Veronese variety over the complex numbers. Indeed, the coordinate ring of the affine cone over the Veronese variety of dimension n and degree d is the ring of invariants $\mathbb{C}[x_0, \ldots, x_n]^{\mu_d}$, where the group $\mu_d \subseteq \mathbb{C}^\times$ of dth roots of 1 acts on each variable by multiplication.

The dualizing sheaf of any projective scheme over a field is S_2 [5, Corollary 5.69]. This fact coupled with the assumption that C is normal and the observation that

ω_C is invertible over $C \setminus P \cong B \setminus E$ implies that ω_C corresponds to the linear equivalence class of a Weil divisor K_C on C. See Schwede's notes [6] for the details of this correspondence.

Assume that the canonical divisor of X is $\mathbb{Q}$-rationally equivalent to a multiple of a hyperplane section $H \subset X$, that is, $K_X \equiv rH$, where $r \in \mathbb{Q}$. In Kollár and Kovács' book [4, Proposition 3.14] it is shown that this assumption is equivalent to the requirement that some multiple of the Weil divisor K_C be Cartier.

Let $H' \subset C$ be a hyperplane section. We have

$$q^*\mathscr{O}_X(1) = p^*\mathscr{O}_C(1)(-E),$$

so passing to divisors we obtain

$$q^*H \equiv p^*H' - E.$$

Using the previously obtained formulas for ω_B in terms of p and q, we conclude

$$\begin{aligned} K_B &\equiv q^*(K_X - H) - 2E \\ &\equiv q^*((r-1)H) - 2E \\ &\equiv p^*((r-1)H') - (r+1)E. \end{aligned}$$

This calculation implies

$$K_C \equiv (r-1)H',$$

so

$$K_B \equiv p^*K_C - (r+1)E.$$

The second term $-(r+1)E$ in the above expression for K_B gives a measure of how bad the singularity at the vertex of C is. If $r+1>0$, sections of (invertible reflexive powers of) ω_C have poles along E if viewed as rational sections of (the corresponding powers of) ω_B. This appearence of poles does not occur in pullbacks of differentials along morphisms of smooth schemes.

Over the complex numbers, the above formula for K_B shows (see [5, Corollary 2.32]) that, in terms of the usual classification of singularities occuring in the minimal model program, we have:

$$\begin{aligned} P \in C \text{ is a terminal sing.} &\Leftrightarrow r < -1 \\ P \in C \text{ is a canonical sing.} &\Leftrightarrow r \leq -1 \\ P \in C \text{ is a klt sing.} &\Leftrightarrow r < 0 \\ P \in C \text{ is a lc sing.} &\Leftrightarrow r \leq 0 \end{aligned}$$

For example, when $X \subseteq \mathbb{P}^N$ is a Veronese variety of dimension n and degree d, we have $\omega_X = \mathscr{O}_{\mathbb{P}^n}(-n-1)$ and $\mathscr{O}_X(1) = \mathscr{O}_{\mathbb{P}^n}(d)$, so that $\omega_X^{\otimes d} = \mathscr{O}_X(-n-1)$ holds in this case and we have $r = -(n+1)/d$. Thus the quotient singularity at the vertex of the cone over a Veronese variety is always klt and will fail to be canonical when $d > n+1$.

If C is not normal (but X is still assumed smooth), the discussion in this section and in the ones that follow still apply provided that we replace C by its normalization and make slight adaptations such as replacing the blow up $p : B \to C$ by its factorization through the normalization of C.

10. Branched covers

Let B be scheme (over the fixed base field). Let L be an invertible sheaf on B. Let

$$U := \operatorname{Spec}_B \operatorname{Sym} L^\vee$$

be the *total space* of L. Let $\pi : U \to B$ denote the projection.

Note that the definition of U using $L^\vee$ as opposed to L has is the correct one, as with it sections of π over an open subset $V \subseteq B$ correspond tautologically to elements of $L(V)$. Indeed:

$$\begin{aligned}\operatorname{Hom}_B(V, U) &= \operatorname{Hom}_{\mathscr{O}_V\text{-algebras}}(\operatorname{Sym} L^\vee|_V, \mathscr{O}_V) \\ &= \operatorname{Hom}_{\mathscr{O}_V\text{-modules}}(L^\vee|_V, \mathscr{O}_V) \\ &= \mathscr{H}om(L^\vee, \mathscr{O}_B)(V) \\ &= L^{\vee\vee}(V) = L(V)\end{aligned}$$

Let $y \in H^0(U, \pi^* L)$ denote the tautological section corresponding to

$$1 \in \mathscr{O}_B = L \otimes L^\vee \subset L \otimes \operatorname{Sym} L^\vee = \pi_* \pi^* L.$$

When L is trivial, $\operatorname{Sym} L^\vee = \mathscr{O}_B[T]$ and y corresponds to the indeterminate T. Thus y can be thought of as a coordinate along the fibers of $\pi : U \to B$.

Let $m \geq 1$ be an integer. Let $s \in H^0(B, L^{\otimes m})$ be a section. Let $Z \subseteq U$ be the subscheme defined by the vanishing of $y^m - s \in \pi^* L^{\otimes m}$. The restriction $\pi|_Z : Z \to B$ and the scheme Z are both called the *(cyclic) branched cover* of B determined by s.

11. Differentials on branched covers

Assume B and Z are smooth. In the next sections we shall give a criterion for when the latter holds. Let us here compute the canonical bundle of Z.

Let $I \subseteq \mathscr{O}_U$ be the ideal sheaf defining Z as a subscheme of U. Because U is smooth, the smoothness of Z is equivalent to the sequence

$$0 \to I/I^2 \to \Omega_U|_Z \to \Omega_Z \to 0$$

being split and exact on the left. As we are assuming that Z is smooth, taking determinants we obtain

$$\begin{aligned}\omega_Z &= \omega_U|_Z \otimes (I/I^2)^\vee \\ &= \omega_U|_Z \otimes \pi^* L^{\otimes m}.\end{aligned}$$

Here we have used the fact that Z is defined as the vanishing inside U of the global section $y^m - s$ of the invertible sheaf $\pi^* L$. This imples that the map

$$\pi^* L^{-m} \xrightarrow{\sim} I$$

given by multiplication by this section is an isomorphism.

To compute ω_U, observe that $\pi : U \to B$ is an affine bundle. This implies that U is locally a product, so the sequence

$$0 \to \pi^* \Omega_B \to \Omega_U \to \Omega_{U/B} \to 0.$$

is exact on the left.

In order to find a description for $\Omega_{U/B}$, consider an arbitrary B-scheme S. Denote its structure map by $s : S \to B$. As remarked above, an S-valued point of the fiber product $U \times_B U$ is a pair of global sections of s^*L. Thus the diagonal $\Delta(U) \subseteq U \times_B U$ is defined by the vanishing of the difference of two sections of

$$\mathrm{pr}_1^* \pi^* L = \mathrm{pr}_2^* \pi^* L.$$

Hence its ideal sheaf is isomorphic to the dual of this line bundle. Pulling the ideal sheaf of a diagonal along its inclusion is a way of constructing sheaves of differentials. Hence the canonical isomorphism

$$\Omega_{U/B} = \pi^* L^\vee.$$

This isomorphism generalizes the fact that for a ring R and indeterminate T, the module of differentials $\Omega_{R[T]/R}$ is freely generated by dT over $R[T]$.

Taking determinants in the exact sequence of differentials of the morphism $\pi : U \to B$, we obtain

$$\omega_U = \pi^*(\omega_B \otimes L^\vee).$$

Putting the above formula for ω_U together with our previous expression for the canonical bundle of Z yields the canonical isomorphism

$$\omega_Z = \pi^*(\omega_B \otimes L^{\otimes(m-1)}).$$

12. Smoothness of branched covers

Keep the setup of the preceding section.

Assume $m > 1$. The only cover with $m = 1$ is the identity.

If the characteristic of the base field is positive, assume that it does not divide the degree m of the branched cover $\pi : Z \to B$. Assume B is smooth. Then Z is smooth if, and only if, the *branching divisor* $\{s = 0\} \subseteq B$ is smooth.

This is a consequence of the following standard result.

Lemma. *Let U be a smooth scheme over a field k. Let $f \in \mathscr{O}_U$ be a global section and let $Z \subseteq U$ be the subscheme defined by the vanishing of f. Then Z is smooth over k if, and only if, the section df of the locally free sheaf $\Omega_{U/S}$ vanishes at no point of Z.*

Because smoothness is local, we may shrink B and assume L is trivial. Then $U = B \times \mathbb{A}^1$. In question is the vanishing of the differential

$$my^{m-1}dy - ds \in \Omega_U = \mathrm{pr}_1^* \Omega_B \oplus \mathscr{O}_U dy$$

along points of Z.

This differential vanishes nowhere along $\{y \neq 0\} \subseteq U$ by the assumption that m is not divisible by the characteristic of the base field. Thus we may restrict attention to points of Z lying in the closed subscheme $\{y = 0\} \subset U$.

This subcheme is naturally identified with B and the restriction of the differential $my^{m-1}dy - ds$ to it is

$$-ds \in \Omega_B \subset \Omega_B \oplus \mathscr{O}_B.$$

The claimed equivalence between the smoothness of Z and that of $\{s = 0\} \subseteq B$ follows easily from this observation.

13. Setup

Let $X \subseteq \mathbb{P}^n$ be a projective variety. Let $C \subseteq \mathbb{P}^{n+1}$ be the projective cone over X. Let $p : B \to C$ be the blow up of the vertex of C. Recall this blow up resolves the natural map $C \dashrightarrow X$ to a morphism $q : B \to X$ and further that q is naturally identified with the projection $\mathbb{P}(\mathscr{O}_X(1) \oplus \mathscr{O}_X) \to X$.

Assume X is smooth and that $K_X \equiv rH$, where $r \in \mathbb{Q}$ and $H \subseteq X$ is a hyperplane section.

Assume for simplicity C is normal, although as remarked above this is not essential. Namely, in the nonnormal case it suffices to replace C with its normalization, p with its factorization through the new C, the one point set $\{P\}$ with its (finite) inverse image in the new C, and all line bundles on C and all sections thereof with their pullbacks to the new C.

Let $d \geq 0$ be an integer and let $L = \mathscr{O}_C(d)$. Let $m \geq 1$ be an integer. If the characteristic of the base field is positive, assume it does not divide m. Let $s \in H^0(C, L^{\otimes m})$ be a section defining a smooth subscheme of C that misses the vertex of this cone.

Let $f : Y \to C$ be the degree m branched cover defined by s. Then Y is smooth away from the m pre-images of the vertex. Form the fiber product:

$$\begin{array}{ccc} Z & \xrightarrow{g} & B \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & C \end{array}$$

Then g is the degree m branched cover of B defined by p^*L and r is a resolution of singularities.

14. Differentials on branched covers of cones

The canonical bundle on Z is given by

$$\omega_Z = \omega_B \otimes L^{m-1},$$

where we omit pullbacks. It follows that

$$g_*(\omega_Z^{\otimes l}) = \oplus_{i=0}^{m-1} \omega_B^{\otimes l} \otimes L^{(m-1)l-i}.$$

Now substitute $L = \mathscr{O}_C(d)$ and

$$\omega_X = \omega_X \otimes \mathscr{O}_X(1) \otimes \mathscr{O}_{CX}(-2)$$

into this equation. The result is

$$g_*(\omega_Z^{\otimes l}) = \oplus_{i=0}^{m-1} \omega_X^{\otimes l} \otimes \mathscr{O}_X(l) \otimes \mathscr{O}_{CX}((b-2)l - di),$$

where $b := d(m-1)$. Using $\mathscr{O}_{CX}(1)(-E) = \mathscr{O}_X(1)$ we conclude

$$g_*(\omega_Z^{\otimes l}) = \oplus_{i=0}^{m-1} \omega_X^{\otimes l} \otimes \mathscr{O}_X((b-1)l - di)([(b-2)l - di]E).$$

Thus, the pluri-canonical forms on the branched cover Y of the given cone are given by

$$\begin{aligned} H^0(Y,\omega_Y^{[l]}) &= H^0(Y\setminus f^{-1}P,\omega_Y^{[l]}) \\ &= H^0(Z\setminus r^{-1}f^{-1}P,\omega_Z^{\otimes l}) \\ &= H^0(B\setminus E, g_*(\omega_Z^{\otimes l})) \\ &= \oplus_{i=0}^{m-1}\oplus_{j=0}^{\infty} H^0(X,\omega_X^{\otimes l}(l(b-1)-di-j)). \end{aligned}$$

On the other hand, the pluri-canonical forms on the resolution Z of this space are given by

$$\begin{aligned} H^0(Z,\omega_Z^{\otimes l}) &= H^0(B, g_*(\omega_Z^{\otimes l})) \\ &= \oplus_{i=0}^{m-1}\oplus_{j=0}^{l(b-2)-di} H^0(X,\omega_X^{\otimes l}(l(b-1)-di-j)). \end{aligned}$$

15. Iitaka dimension computation

Assume l is sufficiently divisible so that rl is an integer and there exists an isomorphism $\omega_X^{\otimes l}\cong \mathscr{O}_X(rl)$. Fix one such.

Note that

$$B_l^Y \le h^0(Y,\omega_Y^{[l]}) \le mB_l^Y$$

where

$$\begin{aligned} B_l^Y &:= \sum_{j=0}^{\infty} h^0(X,\mathscr{O}_X((r+b-1)l-j)) \\ &= \sum_{j=0}^{\lfloor (r+b-1)l\rfloor} h^0(X,\mathscr{O}_X((r+b-1)l-j)). \end{aligned}$$

This says in particular that $B_l^Y=0$ when $r+b-1<0$. When $r+b-1=0$, we have a unique nonzero summand and

$$B_l^Y = h^0(X,\mathscr{O}_X)=1.$$

To study the case when $r+b-1>0$, we set

$$B_l := \sum_{j=0}^{l} h^0(X,\mathscr{O}_X((r+b-1)l-j)).$$

Then

$$B_l \le B_l^Y \le \lceil r+b-1\rceil B_l.$$

Let $N\in\mathbb{Z}$ be a positive integer such that $r+b-1-1/N>0$ and assume that l, in addition to being divisible as above, is also such that l/N is an integer. Then

$$\begin{aligned} (l/N)h^0(X,\mathscr{O}_X((r+b-1-1/N)l)) &\le \sum_{j=0}^{l/N} h^0(X,\omega_X^{\otimes l}((b-1)l-j)) \\ &\le B_l \\ &\le l h^0(X,\mathscr{O}_X((r+b-1)l). \end{aligned}$$

We conclude the following, where κ denotes Iitaka dimension:

$$r > 1 - b \Rightarrow \kappa(\omega_Y) = \dim X + 1$$
$$r = 1 - b \Rightarrow \kappa(\omega_Y) = 0$$
$$r < 1 - b \Rightarrow \kappa(\omega_Y) = -\infty$$

16. Kodaira dimension computation

Consider now the resolution Z of Y obtained by blowing up the m inverse images of the vertex of the cone over X. We again have

$$B_l^Z \leq h^0(Z, \omega_Z^{\otimes l}) \leq m B_l^Y$$

where

$$B_l^Z := \sum_{j=0}^{(b-2)l} h^0(X, \mathscr{O}_X((r+b-1)l - j))$$

Thus if $b = 0$ or $b = 1$, we have $B_l^Z = 0$, so

$$b \leq 1 \Rightarrow \kappa(\omega_Z) = -\infty.$$

Note that Z is covered by rational curves in these cases. Indeed, when $b = 0$, the cover $Z \to B$ is either the identity (when $m = 1$) or Z is the disjoint uniton union of copies of B (when $d = 0$). When $b = 1$, the map $g : Z \to B$ is a double cover branched along a quadric. This implies that g restricts to double covers branched at two points over each of rays of the cone. Such double covers are rational curves.

If $b = 2$, the sum defining B_l^Z degenerates to its $j = 0$ term, so

$$b = 2 \text{ and } r > -1 \Rightarrow \kappa(\omega_Z) = \dim X$$
$$b = 2 \text{ and } r = -1 \Rightarrow \kappa(\omega_Z) = 0$$
$$b = 2 \text{ and } r < -1 \Rightarrow \kappa(\omega_Z) = -\infty.$$

Note that the -1 that appears here is the same as the $1 - b$ that we had before.

Note that $b = 2$ corresponds to either triple covers branched over a cubic or double covers branched over quartic.

Finally, if $b > 2$, then we have

$$B_l \leq B_l^Z \leq (b-2) B_l$$

and as before we conclude

$$b > 2 \text{ and } r > 1 - b \Rightarrow \kappa(\omega_Z) = \dim X + 1$$
$$b > 2 \text{ and } r = 1 - b \Rightarrow \kappa(\omega_Z) = 0$$
$$b > 2 \text{ and } r < 1 - b \Rightarrow \kappa(\omega_Z) = -\infty.$$

In summary, we have

$$\kappa(\omega_Y) \neq \kappa(\omega_Z)$$

if, and only if,

$$b \leq 1 \text{ and } r \geq 1 - b \geq 0$$

or

$$b = 2 \text{ and } r > 1 - b = -1.$$

References

[1] David Cox, John Little, and Henry Schenck. *Toric Varieties*. Graduate Studies in Mathematics. American Mathematical Society, 2011.

[2] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton.

[3] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977.

[4] J. Kollár and S. Kovács. *Singularities of the Minimal Model Program*. Cambridge Tracts in Mathematics. Cambridge University Press, 2013.

[5] J. Kollár and S. Mori. *Birational Geometry of Algebraic Varieties*. Cambridge Tracts in Mathematics. Cambridge University Press, 2008.

[6] Karl Schwede. Generalized divisors and reflexive sheaves. *Unpublished*. `http://www.math.utah.edu/~schwede/Notes/GeneralizedDivisors.pdf` (Retrieved on February 22, 2016).