A SPANNING TREE INVARIANT FOR MARKOV SHIFTS

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Abstract. We introduce a new type of invariant of block isomorphism for Markov shifts, defined by summing the weights of all spanning trees for a presentation of the Markov shift. We give two proofs of invariance. The first uses the Matrix-Tree Theorem to show that this invariant can be computed from a known invariant, the stochastic zeta function of the shift. The second uses directly the definition to show invariance under state splitting, from which all block isomorphisms can be built.

Key words. Markov shift, block isomorphism, spanning tree, Matrix-Tree Theorem.

AMS(MOS) subject classifications. Primary: 37A35, 37A50, 37B10, 60J10.

1. Introduction. Invariants of dynamical systems typically make use of recurrent or asymptotic behavior. Examples include entropy, mixing, and periodic points. Here we define a quantity for stochastic Markov shifts that is invariant under block isomorphism, and which has a different flavor. For a given presentation of the Markov shift, we add up the weights of all spanning trees for the graph. Since spanning trees are maximal subgraphs without loops, this is in some sense an operation that is orthogonal to recurrent behavior.

We prove invariance of the spanning tree quantity under block isomorphism in two ways. The first shows that it can be computed from the stochastic zeta function of the Markov shift, an invariant introduced in [4]. The second is a more "barehands" structural approach, using only the definition to show that it is invariant under the elementary block isomorphisms corresponding to state splittings.

2. The Matrix-Tree Theorem. In this section we give a brief account of the Matrix-Tree Theorem for directed graphs. See [1, II.3] for more details.

Let G be a (finite, directed) graph. We suppose that the vertex set of G is $\mathcal{V} = \{1, 2, \dots, v\}$. We sometimes call vertices *states*. Let $\mathcal{E}$ be the edge set of G. Denote the subset of edges from state i to state j by $\mathcal{E}_i^j$. Put $\mathcal{E}_i = \bigcup_j \mathcal{E}_i^j$, the set of all edges starting at state i, and $\mathcal{E}^j = \bigcup_i \mathcal{E}_i^j$, the set of all edges ending at state j.

A *tree in G rooted at $r \in \mathcal{V}$* is a subgraph T of G such that every vertex in T except r has a unique outgoing edge in T, there is no outgoing edge in T at r, and from every vertex in T except r there is a unique path ending at r. See Figure 1(a). We abbreviate this by saying that T is a *tree in G at r*. A tree is *spanning* if it contains every state. Let $\mathcal{S}_r$ denote the set of spanning trees at r, and $\mathcal{S} = \bigcup_r \mathcal{S}_r$ be the set of all spanning trees in G.

Consider the elements of $\mathcal{E}$ to be commuting abstract variables, and form the ring $\mathbb{Z}[\mathcal{E}]$ of polynomials in the variables from $\mathcal{E}$ with integer coefficients. For any subgraph H of G define the *weight* of H to be $\prod_{e \in H} e \in \mathbb{Z}[\mathcal{E}]$, where the product

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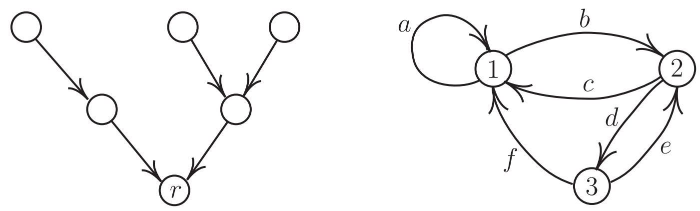

(a) (b)

FIG. 1. *A typical tree at r, and a graph*

is over the edges in H. For a subset $\mathcal{F} \subset \mathcal{E}$ put $\Sigma(\mathcal{F}) = \sum_{e\in\mathcal{F}} e \in \mathbb{Z}[\mathcal{E}]$. The *Kirchhoff matrix* K of G is the $v \times v$ matrix $K = [K_{ij}]$ defined by

$$K_{ij} = \Sigma(\mathcal{E}_i)\delta_{ij} - \Sigma(\mathcal{E}_i^j),$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Notice that no self-loops occur in K. Let $K^{(r)}$ denote the rth principal minor of K, that is the determinant of the matrix formed by removing the rth row and rth column from K. Let adj K be the adjoint matrix of K, and let tr denote the trace of a matrix.

THEOREM 2.1 (Matrix-Tree Threorem [1, II.3]). *Using the notations above,*

$$\sum_{S\in\mathcal{S}_r} w(S) = K^{(r)}, \quad \textit{and so} \quad \sum_{S\in\mathcal{S}} w(S) = \operatorname{tr}[\operatorname{adj} K]$$

EXAMPLE 1. For the graph in Figure 1(b),

$$K = \begin{bmatrix} b & -b & 0 \\ -c & c+d & -d \\ -f & -e & e+f \end{bmatrix}.$$

Then $K^{(1)} = ce + cf + df$ enumerates the spanning trees at 1, and similarly for $K^{(2)} = be + bf$ and $K^{(3)} = bd$.

3. Markov shifts. Let $P = [p_{ij}]$ be a $v \times v$ stochastic matrix, so that $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for every i. We assume from now on that P is irreducible. Let $G(P)$ be the directed graph with vertex set $\mathcal{V} = \{1, \dots, v\}$, and with exactly one edge from state i to state j if $p_{ij} > 0$, and no such edge if $p_{ij} = 0$. Let $\mathcal{E}$ denote the resulting edge set for $G(P)$.

The *shift of finite type* determined by $G(P)$ is the subset $X_{G(P)}$ of $\mathcal{E}^{\mathbb{Z}}$ defined by

$$X_{G(P)} = \{\dots e_{-1}e_0e_1 \dots \in \mathcal{E}^{\mathbb{Z}} : e_{n+1} \text{ follows } e_n \text{ in } G(P)\}.$$

See [2, Chap. 2] for further details.

By the irreducibility assumption, there is a unique Markov probability measure μ_P on $X_{G(P)}$ with transition probabilities p_{ij}. Let σ_P denote the left shift on

$X_{G(P)}$, so that μ_P is σ_P-invariant. The measure-preserving system $(X_{G(P)}, \mu_P, \sigma_P)$ is the *Markov shift determined by* P.

Let Q be another stochastic matrix, of possibly different dimension. We say that the Markov shifts determined by P and by Q are *block isomorphic* if there is a shift-commuting measure-preserving homeomorphism between them. In other words, a block isomorphism from $(X_{G(P)}, \mu_P, \sigma_P)$ to $(X_{G(Q)}, \mu_Q, \sigma_Q)$ is a homeomorphism $\psi\colon X_{G(P)} \to X_{G(Q)}$ such that $\sigma_Q \circ \psi = \psi \circ \sigma_P$ and $\mu_Q = \mu_P \circ \psi^{-1}$.

4. The spanning tree invariant. As in the previous section, let $P = [p_{ij}]$ be an irreducible stochastic matrix and $G = G(P)$ be its associated directed graph. If $e \in \mathcal{E}$ goes from i to j put $p(e) = p_{ij}$. For any subgraph H of G define the P*-weight* (or simply the *weight*) of H to be $w_P(H) = \prod_{e\in H} p(e)$.

DEFINITION 4.1. *Let P be an irreducible stochastic matrix, and let $\mathcal{S}$ denote the set of spanning trees for $G(P)$. Define the* spanning tree invariant *of P to be*

$$\tau(P) = \sum_{S\in\mathcal{S}} w_P(S).$$

EXAMPLE 2. (1) If $P = \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix}$ then $\tau(P) = 1 - p + q$. In particular, if $p = q = 1/2$ then $\tau(P) = 1$.

(2) If

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \quad \text{then } \tau(P) = \frac{9}{4}.$$

Note that there is a uniformly three-to-one measure-preserving factor map from this Markov shift onto the Bernoulli shift in part (1) with $p = q = 1/2$. Thus τ is not in general preserved by such factor maps.

To justify its name, we will prove that τ is invariant under block isomorphism.

THEOREM 4.1. *If P and Q are irreducible stochastic matrices whose associated Markov shifts are block isomorphic, then $\tau(P) = \tau(Q)$.*

We will give two proofs of invariance. The first computes $\tau(P)$ in terms of a known invariant, the stochastic zeta function of P. The second is more “structural,” showing proving invariance of τ for each of the basic building blocks of a block isomorphism.

5. First proof of invariance. Define $\phi_P\colon \mathbb{Z}[\mathcal{E}] \to \mathbb{R}$ on the variables e by $\phi_P(e) = p(e)$, and extend it to a ring homomorphism. Applying ϕ_P to the Matrix-Tree Theorem for $G = G(P)$ gives

$$\begin{aligned}\tau(P) &= \sum_{S\in\mathcal{S}} w_P(S) = \sum_{S\in\mathcal{S}} \phi_P\big(w(S)\big) = \phi_P\Big(\sum_{S\in\mathcal{S}} w(S)\Big)\\ &= \phi_P\big(\operatorname{tr}[\operatorname{adj} K]\big) = \operatorname{tr}\big[\operatorname{adj} \phi_P(K)\big].\end{aligned}$$

Now $\phi_P(K) = I - P$ since P is stochastic. Hence $\tau(P) = \mathrm{tr}\big[\mathrm{adj}(I - P)\big]$.

Let the eigenvalues of P be $\lambda_1 = 1, \lambda_2, \dots, \lambda_v$, where $\lambda_j \neq 1$ and $|\lambda_j| \leq 1$ for $2 \leq j \leq v$. Since formation of the adjoint commutes with conjugation and trace is invariant under conjugation, conjugating P to its Jordan form shows that

$$\tau(P) = \mathrm{tr}\big[\mathrm{adj}(I - P)\big] = \prod_{j=2}^{v}(1 - \lambda_j). \tag{5.1}$$

Recall the *stochastic zeta function* $\zeta_P(t)$ of P, defined in [4] as

$$\zeta_P(t) = \exp\Big[\sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{C \in \mathcal{C}_n} w_P(C)\Big],$$

where $\mathcal{C}_n$ is the set of all cycles in $G(P)$ of length n. The stochastic zeta function is invariant under block isomorphism. It can be computed in terms of P as

$$\zeta_P(t) = \frac{1}{\det[I - tP]}.$$

Hence

$$(1/\zeta_P)(t) = \det[I - tP] = \prod_{k=1}^{v}(1 - \lambda_k t),$$

so that

$$(1/\zeta_P)'(t) = \sum_{k=1}^{v} -\lambda_k \prod_{j \neq k}(1 - \lambda_j t).$$

Thus

$$(1/\zeta_P)'(1) = -\prod_{j=2}^{v}(1 - \lambda_j) = -\tau(P).$$

This shows that $\tau(P)$ can be computed from ζ_P, and hence is an invariant of block isomorphism.

6. Invariance under in-splitting. Every block isomorphism between Markov shifts is a composition of basic block isomorphisms obtained from state splitting and permuting states. This was a fundamental discovery of R. Williams [5]. For further background on state splitting and the decomposition of block isomorphisms, the reader is referred to [4] as well as §2.4 and Theorem 7.1.2 of [2]. Permuting states clearly preserves τ, so we focus on the behavior of τ under state splitting.

Let k be a fixed state in $G = G(P)$. There are two types of state splitting at k: in-splitting from a partition of the incoming edges to k, and out-splitting from a partition of the outgoing edges from k. These are handled by separate arguments, in-splitting in this section and out-splitting in the next. As might be expected from the directional nature of shifts of finite type, in-splitting is easier to handle that out-splitting.

It is sufficient, as well as notationally simpler, to consider in-splitting k into just two states. For this we partition $\mathcal{E}^k$ into the sets $\mathcal{F}_1$ and $\mathcal{F}_2$. Form a new graph G' as follows. Replace state k with two new states k_1 and k_2. Every edge in G from k to $j \neq k$ is duplicated as two edges in G', one from k_1 to j and one from k_2 to j. An edge f from i to k lies in either $\mathcal{F}_1$ or $\mathcal{F}_2$. If $f \in \mathcal{F}_1$, then in G' put a corresponding edge from i to k_1 and no edge from i to k_2 (if $i = k$, then in G' there are edges from both k_1 and k_2 to k_1); similarly if $f \in \mathcal{F}_2$. Figure 2 depicts such an in-splitting.

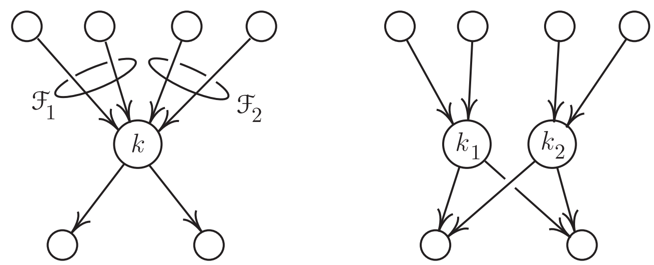

FIG. 2. *In-splitting a state*

Preservation of measure shows that under such an in-splitting the transition matrix P becomes P' on G' defined as follows. For notational convenience we use $p'(i,j)$ instead of p'_{ij}. If $i, j \neq k_m$ $(m = 1, 2)$ then $p'(i,j) = p(i,j)$. If $i \neq k_1, k_2$ and the edge from i to k is in $\mathcal{F}_m$ then $p'(i, k_m) = p(i,k)$. Finally, if the edge from k to k is in $\mathcal{F}_m$, then $p'(k_n, k_m) = p(k,k)$ for $n = 1, 2$. For example, if $k = 1$, $\mathcal{F}_1 = \{1, \dots, \ell\}$, and $\mathcal{F}_2 = \{\ell+1, \dots, v\}$, then P' has the form

$$P' = \begin{bmatrix} p_{11} & 0 & p_{12} & \cdots & p_{1v} \\ p_{11} & 0 & p_{12} & \cdots & p_{1v} \\ p_{21} & 0 & p_{22} & \cdots & p_{2v} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{\ell 1} & 0 & p_{\ell 2} & \cdots & p_{\ell v} \\ 0 & p_{\ell+1,1} & p_{\ell+1,2} & \cdots & p_{\ell+1,v} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & p_{v1} & p_{v2} & \cdots & p_{vv} \end{bmatrix}.$$

We next construct a correspondence between certain sets of spanning trees in G and similar sets in G'. Consider a triple of the form (T, T_1, T_2), where T is a tree in G at some vertex r, each T_m is a tree in G at k using only edges from $\mathcal{F}_m$ $(m = 1, 2)$, the three trees are disjoint except for the common vertex k, and they span all vertices of G. We specifically allow the possibility of the empty tree (with no vertices or edges), and also the tree consisting of a single vertex and no edges. In particular, if $k = r$ then T is empty.

Each such triple (T, T_1, T_2) in G corresponds to a triple (T', T_1', T_2') in G', where T is copied over to T' verbatim, and T_m' is the tree at k_m obtained from T_m $(m = 1, 2)$. Figure 3 illustrates this correspondence. This triple has the property that T' is a tree in G' at r, T_m' is a tree in G' at k_m for $m = 1, 2$, all three trees are disjoint, and they span the vertices of G'. There is clearly a one-to-one correspondence between the set of such triples in G and those in G'.

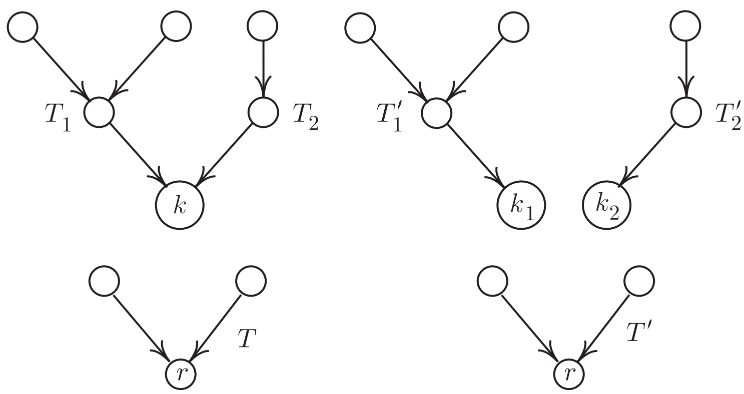

FIG. 3. *Correspondence of trees under an in-splitting*

Let $\mathcal{S}(T, T_1, T_2)$ be the set of all spanning trees in G at r containing T, T_1, and T_2. Similarly define $\mathcal{S}(T', T_1', T_2')$ in G'. Clearly the set $\mathcal{S}(G)$ of spanning trees in G is the disjoint union of the $\mathcal{S}(T, T_1, T_2)$ over all possible triples, and similarly $\mathcal{S}(G')$ is the disjoint union of the $\mathcal{S}(T', T_1', T_2')$. Hence to prove that $\tau(P) = \tau(P')$, it suffices to show that

$$\sum_{S \in \mathcal{S}(T, T_1, T_2)} w_P(S) = \sum_{S' \in \mathcal{S}(T', T_1', T_2')} w_{P'}(S'). \tag{6.1}$$

Fix a triple (T, T_1, T_2). The only way to create a spanning tree in G at r containing these trees is to add an edge from k to T. Thus if $p(k, T)$ denotes the sum of the transition probabilities from k to the vertices of T, it follows that

$$\sum_{S \in \mathcal{S}(T, T_1, T_2)} w_P(S) = w_P(T) w_P(T_1) w_P(T_2) p(k, T).$$

Consider the corresponding triple (T', T_1', T_2') in G'. There are now three ways to form a spanning tree at r in G' containing these trees: (1) join k_1 to T_2' and k_2 to T', (2) join k_2 to T_1' and k_1 to T', and (3) join both k_1 and k_2 to T'. The contribution of adding these two edges to the total weight is, respectively,

$p'(k_1, T_2')p'(k_2, T')$, $p'(k_2, T_1')p'(k_1, T')$, and $p'(k_1, T')p'(k_2, T')$. Hence

$$\sum_{S' \in \mathcal{S}(T', T_1', T_2')} w_{P'}(S') = w_{P'}(T')w_{P'}(T_1')w_{P'}(T_2') \times \big[p'(k_1, T_2')p'(k_2, T') + p'(k_2, T_1')p'(k_1, T') + p'(k_1, T')p'(k_2, T')\big]$$

Let us assume that if there is an edge from k to itself in G, then this edge lies is $\mathcal{F}_1$. Now $w_P(T) = w_{P'}(T')$, $w_P(T_1) = w_{P'}(T_1')$, and $w_P(T_2) = w_{P'}(T_2')$. Furthermore, $p'(k_1, T') = p'(k_2, T') = p(k, T)$, and $p'(k_1, T_2) = p(k, T_2) - p(k, k)$, $p'(k_2, T_1) = p(k, T_1)$. Hence

$$\begin{aligned} &p'(k_1, T_2')p'(k_2, T') + p'(k_2, T_1')p'(k_1, T') + p'(k_1, T')p'(k_2, T') \\ &= p(k, T)\big[p(k, T_1) + p(k, T_2) - p(k, k) + p(k, T)\big] = p(k, T) \end{aligned}$$

since T_1 and T_2 are disjoint except for the common vertex k. This proves (6.1), and completes the proof that τ is invariant under in-splitting.

7. Invariance under out-splitting. To consider out-splittings, fix a state k in G. Partition the set $\mathcal{E}_k$ of outgoing edges from k into two sets $\mathcal{F}_1$ and $\mathcal{F}_2$. Form the out-split graph G' as follows. Replace k with two new states k_1 and k_2. Each incoming edge from a state $i \neq k$ to k is duplicated to two edges, one from i to k_1 and one from i to k_2. An edge $f \in \mathcal{F}_1$ from k to j induces a corresponding edge from k_1 to j in G' (if $j = k$, then include edges from k_1 to both k_1 and k_2), and similarly for $\mathcal{F}_2$. Figure 4 depicts a typical out-splitting at k.

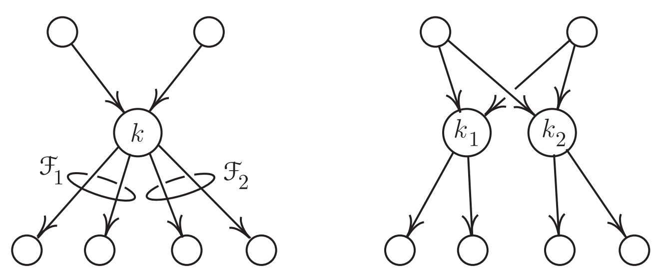

FIG. 4. *Out-splitting a state*

The matrix P' on G' corresponding to P is defined as follows. Let $q = \sum_{e \in \mathcal{F}_1} p(e)$, so that $1 - q = \sum_{e \in \mathcal{F}_2} p(e)$. If $i, j \neq k_m$ put $p'(i, j) = p(i, j)$. If $j \neq k_m$ put $p'(k_1, j) = p(k, j)/q$ and $p'(k_2, j) = p(k, j)/(1 - q)$. If $i \neq k_m$ put $p'(i, k_1) = q\, p(i, k)$ and $p'(i, k_2) = (1 - q)p(i, k)$. Finally, if there is a loop at k, assume that it is contained in $\mathcal{F}_1$ (the alternative case is similar). Then put $p'(k_1, k_1) = q\, p(k, k)/q = p(k, k)$ and $p'(k_1, k_2) = (1 - q)p(k, k)/q$. For example, if $k = 1$, $\mathcal{F}_1 = \{1, 2, \dots, \ell\}$, and $\mathcal{F}_2 = \{\ell+1, \dots, v\}$, then

$$P' = \begin{bmatrix} \frac{q}{q} p_{11} & \frac{1-q}{q} p_{11} & \frac{1}{q} p_{12} & \dots & \frac{1}{q} p_{1\ell} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{p_{1,\ell+1}}{1-q} & \dots \frac{p_{1v}}{1-q} & \\ q\, p_{21} & (1-q) p_{21} & p_{22} & \dots & p_{2\ell} & p_{2,\ell+1} & \dots & p_{2v} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ q\, p_{v1} & (1-q) p_{v1} & p_{v2} & \dots & p_{v\ell} & p_{v,\ell+1} & \dots & p_{vv} \end{bmatrix}.$$

Next, consider pairs (T, U) of subgraphs of G such that T is a tree at some vertex r, U is a tree at k, and T and U are disjoint and contain all vertices of G. For each such pair (T, U) let $\mathcal{B}$ denote the set of immediate predecessor states of k in U, so that $i \in \mathcal{B}$ if and only if the edge from i to k is in U. Each subset $B \subset \mathcal{B}$ induces two subtrees $U_1(B)$ and $U_2(B)$ rooted at k and which together span U, where $U_1(B)$ is the subtree of U including all predecessors in U of states in B, and $U_2(B)$ is defined similarly using $B^c = \mathcal{B} \setminus B$.

Each $B \subset \mathcal{B}$ then yields a triple $\big(T', U_1'(B), U_2'(B)\big)$ in G', where T' is copied directly from T, $U_1'(B)$ is the tree in G' at k_1 using the edges of $U_1(B)$, and $U_2'(B)$ is the tree in G' at k_2 using the edges of $U_2(B)$. Thus each pair (T, U) corresponds to the collection of triples $\{\big(T', U_1'(B), U_2'(B)\big) : B \subset \mathcal{B}\}$. Figure 5 illustrates this construction.

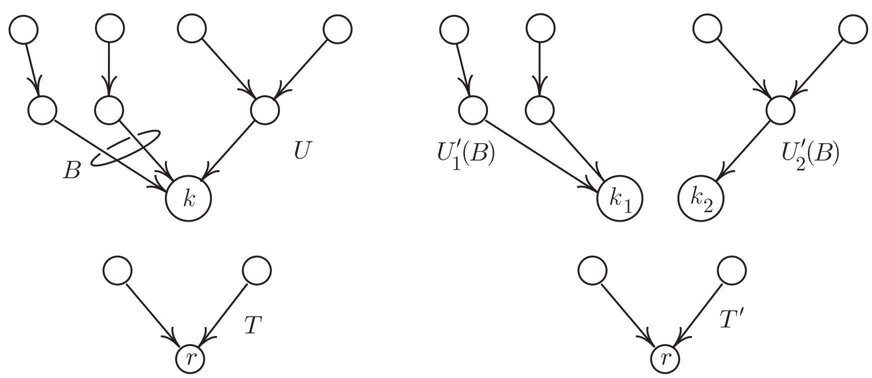

FIG. 5. *Correspondence of trees under an out-splitting*

Let $\mathcal{S}(T, U)$ denote the set of spanning trees in G at r containing T and U, and $\mathcal{S}\big(T', U_1'(B), U_2'(B)\big)$ be the set of spanning trees in G' containing T', $U_1'(B)$, and $U_2'(B)$. Then $\mathcal{S}(G)$ is the disjoint union of the $\mathcal{S}(T, U)$ and $\mathcal{S}(G')$ is the disjoint union of the $\mathcal{S}\big(T', U_1'(B), U_2'(B)\big)$. Therefore it suffices to show that for each pair (T, U) we have that

$$\sum_{S \in \mathcal{S}(T,U)} w_P(S) = \sum_{B \subset \mathcal{B}} \sum_{S' \in \mathcal{S}(T', U_1'(B), U_2'(B))} w_{P'}(S'). \tag{7.1}$$

Fix a pair (T, U). Let

$$a_1 = p(\mathcal{F}_1, T) = \sum\{p(e) : e \in \mathcal{F}_1 \text{ and } e \text{ terminates in } T\},$$

and similarly $b_1 = p(\mathcal{F}_1, U)$, $a_2 = p(\mathcal{F}_2, T)$, and $b_2 = p(\mathcal{F}_2, U)$. Then $a_1+b_1 = q$ and $a_2 + b_2 = 1 - q$.

The only additional edge needed to form a spanning tree from (T, U) is an edge from k to T. Hence

$$\sum_{S\in\mathcal{S}(T,U)} w_P(S) = w_P(T)w_P(U)p(k, T) = w_P(T)w_P(U)[a_1 + a_2].$$

Next, let $B \subset \mathcal{B}$, and form the triple $\big(T', U_1'(B), U_2'(B)\big)$. There are now three ways to form a spanning tree in $\mathcal{S}\big(T', U_1'(B), U_2'(B)\big)$: (1) join k_1 to $U_2'(B)$ and k_2 to T', (2) join k_2 to $U_1'(B)$ and k_1 to T', and (3) join both k_1 and k_2 to T'. Hence

$$\sum_{S'\in\mathcal{S}\left(T',U_1'(B),U_2'(B)\right)} w_{P'}(S') = w_{P'}(T')w_{P'}(U_1'(B))w_{P'}(U_2'(B))\Phi(B),$$

where

$$\begin{aligned}\Phi(B) = p'(\mathcal{F}_1, U_2'(B))p'(k_2, T') + p'(\mathcal{F}_2, U_1'(B))p'(k_1, T')\\ + p'(k_1, T')p'(k_2, T').\end{aligned}$$

Note that $w_{P'}(T') = w_P(T)$. Let $n = |\mathcal{B}|$. Since $U_1'(B)$ uses $|B|$ incoming edges each of whose weight has been multiplied by the factor q, and $U_2'(B)$ uses $n - |B|$ edges each of whose weight is multiplied by a factor $1 - q$, we have that

$$w_{P'}(U_1'(B))w_{P'}(U_2'(B)) = q^{|B|}(1 - q)^{n-|B|}w_P(U).$$

Cancelling the common term $w_P(T)w_P(U)$ reduces (7.1) to proving that

$$a_1 + a_2 = \sum_{B\subset\mathcal{B}} q^{|B|}(1 - q)^{n-|B|}\Phi(B). \tag{7.2}$$

Now $p'(k_1, T') = a_1/q$ and $p'(k_2, T') = a_2/(1 - q)$. Let $\mathcal{F}_1(U)$ denote the set of edges in $\mathcal{F}_1$ ending in U. Then by interchanging the order of summation see that

$$\begin{aligned}\sum_{B\subset\mathcal{B}} q^{|B|}(1 - q)^{n-|B|}p'(\mathcal{F}_1, U_2'(B)) &= \sum_{e\in\mathcal{F}_1(U)} p'(e) \sum_{e\in B^c} q^{|B|}(1 - q)^{n-|B|}\\ &= \sum_{e\in\mathcal{F}_1(U)} \frac{p(e)}{q} \sum_{k=0}^{n-1} \binom{n-1}{k} q^k(1 - q)^{n-k}\\ &= \frac{1-q}{q} \sum_{e\in\mathcal{F}_1(U)} p(e) = \frac{1-q}{q} b_1.\end{aligned}$$

Similarly,

$$\sum_{B\subset\mathcal{B}} q^{|B|}(1-q)^{n-|B|}p'(\mathcal{F}_2, U_1'(B)) = \frac{q}{1-q}b_2.$$

Since $b_1 = 1 - a_1$ and $b_2 = 1 - q - a_2$, we obtain that

$$\begin{aligned}\sum_{B\subset\mathcal{B}} q^{|B|}(1-q)^{n-|B|}\Phi(B) &= \frac{1-q}{q}b_1\frac{a_2}{1-q} + \frac{q}{1-q}b_2\frac{a_1}{q} + \frac{a_1}{q}\frac{a_2}{1-q} \\ &= \frac{a_2(q-a_1)}{q} + \frac{a_1(1-q-a_2)}{1-q} + \frac{a_1a_2}{q(1-q)} \\ &= a_1 + a_2.\end{aligned}$$

This establishes (7.2), and completes the proof.

8. Concluding remarks. (1) The possibility of using spanning trees to define an invariant was first observed experimentally using *Mathematica*.

(2) It is possible to obtain finer invariants by use of the matrix of powers $P^t = [p_{ij}^t]$ as in [3].

(3) If P is $v \times v$, then (5.1) shows that $\tau(P) \le 2^{v-1}$. Thus $1 + \log_2 \tau(P)$ is a lower bound on the size of any irreducible Markov shift that is block isomorphic to P.

(4) Using elementary matrix operations, one can show directly that $\tau(P) = \tau(P')$, where P' is derived from P using in-splitting or out-splitting as above. This shows that τ is an invariant of block isomorphism without use of the stochastic zeta function.

(5) Graphs with positive weights can be interpreted as electrical resistance networks, and the use of spanning trees to compute total resistance goes back to Kirchhoff. It may be possible to use ideas from electrical networks to find other invariants of Markov shifts.

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