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Small polynomial matrix presentations of nonnegative matrices

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Abstract

We investigate the use of polynomial matrices to give efficient presentations of nonnegative matrices exhibiting prescribed spectral and algebraic behavior.

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1. Introduction

Let $\mathbb{S}$ be a unital subring of the real numbers $\mathbb{R}$, and $\mathbb{S}_+$ denote the set of its nonnegative elements. The inverse spectral problem for nonnegative matrices asks for necessary and sufficient conditions on an n-tuple of complex numbers for it to be the spectrum of an $n \times n$ matrix over $\mathbb{S}_+$. When $\mathbb{S} = \mathbb{R}$ various ingenious and fascinating partial results are known (see results, discussions, and references in [2,12,20,21] and more recently [15,16]). There is a clear conjectural characterization in [6] of which lists of nonzero complex numbers can be the nonzero part of the spectrum of a matrix over $\mathbb{S}_+$. This conjecture has been verified for many $\mathbb{S}$, including the main cases $\mathbb{S} = \mathbb{R}$ [6] and $\mathbb{S} = \mathbb{Z}$ [13], but the problem of determining reasonable upper

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bounds for the minimum size of a matrix with a given nonzero spectrum is still out of reach, even for $\mathbb{S} = \mathbb{R}$.

The use of matrices whose entries are polynomials with nonnegative coefficients to represent nonnegative matrices goes back at least to the original work of Shannon on information theory [24, Section 1]. Such matrices can provide much more compact presentations of nonnegative matrices exhibiting prescribed phenomena, as well as give a more amenable and natural algebraic framework [4], of particular value in symbolic dynamics [5]. Their use focuses attention naturally on asymptotic behavior having a comprehensible theory. In particular, it seems to us that the problem of determining the minimum size polynomial matrix presenting a given nonzero spectrum is likely to have a satisfactory and eventually accessible solution, which may also be useful for bounding the size of nonpolynomial matrix presentations.

In this paper we give realization results, constructing polynomial matrices of small size presenting nonnegative matrices satisfying certain spectral and algebraic constraints. Perhaps the main contribution is to show how certain geometrical ideas interact with polynomial matrices. We hope that the combined geometric-polynomial viewpoint may be useful in approaching deeper problems. For example, the minimum size problem and the Generalized Spectral Conjecture [5,7] may be approached in terms of turning the epimorphisms of Theorems 5.1 and 8.8 into isomorphisms.

For the statement of our specific results, recall a matrix is *primitive* if it is nonnegative and some power is strictly positive. The inverse spectral problem for nonnegative matrices reduces to the inverse spectral problem for primitive matrices [6]. The Perron theorem shows that one necessary condition on a list Λ of complex numbers for it to be the spectrum of a primitive matrix is that there be one positive element, called the *spectral radius of* Λ, that is strictly larger than the absolute value of each of the other elements. If one further requires that Λ be the spectrum of a primitive matrix over $\mathbb{S}$, then Λ must also be $\mathbb{S}$-*algebraic*, that is, the monic polynomial whose roots are the elements of Λ must have coefficients in $\mathbb{S}$.

In Section 3 we show how to associate naturally to each matrix with entries in $\mathbb{S}_+[t]$ a corresponding matrix with entries in $\mathbb{S}_+$. Handelman [9] showed that an $\mathbb{S}$-algebraic list Λ satisfying the Perron condition above is contained in the spectrum of a primitive matrix over $\mathbb{S}_+$ with the same spectral radius corresponding to a 1×1 polynomial matrix if and only if no other element of Λ is a positive real number. After developing some machinery for polynomial matrices in Sections 3 and 4, we show that *every* $\mathbb{S}$-algebraic Λ satisfying the Perron condition is contained in the spectrum of a primitive matrix with the same spectral radius coming from a 2×2 polynomial matrix over $\mathbb{S}_+[t]$. This answers a question raised in [4, Section 5.9] and generalizes a result of Perrin (see Remark 6.7). The proof, combined with a simple geometrical observation, allows us to recover Handelman's original result in Section 7. In Section 8 we refine our results for nonzero spectra by finding small polynomial matrix presentations for actions on appropriate $\mathbb{S}$-modules.

We thank Robert Mouat for suggesting an important simplification in the basic construction of Section 3.

2. Preliminaries

We collect here some convenient notation and terminology.

Let $\mathbb{S}$ denote an arbitrary unital subring of the reals $\mathbb{R}$, so that $\mathbb{S}$ is a subring containing 1. Note that $\mathbb{S}$ is either the discrete subring $\mathbb{Z}$ of integers or is dense in $\mathbb{R}$. Denote by $\mathbb{K}$ the quotient field of $\mathbb{S}$. We let $\mathbb{S}_+ = \mathbb{S} \cap [0, \infty)$ denote the nonnegative semiring of $\mathbb{S}$, and $\mathbb{S}_{++} = \mathbb{S} \cap (0, \infty)$ be the set of strictly positive elements of $\mathbb{S}$. The ring of polynomials with coefficients in $\mathbb{S}$ is denoted by $\mathbb{S}[t]$, and the semiring of polynomials with coefficients in $\mathbb{S}_+$ by $\mathbb{S}_+[t]$.

A *list* is a collection of complex numbers where multiplicity matters but order does not. We use the notation $\Lambda = \langle \lambda_1, \ldots, \lambda_n \rangle$ for a list, so that $\langle 1, 1, 2 \rangle = \langle 2, 1, 1 \rangle \neq \langle 1, 2 \rangle$. A list Λ is *contained in* another list Λ', in symbols $\Lambda \subset \Lambda'$, if for every $\lambda \in \Lambda$ the multiplicity of λ in Λ is less than or equal to its multiplicity in Λ'.

The *spectral radius* of a list Λ is the number $\rho(\Lambda) = \max_{\lambda \in \Lambda} |\lambda|$. A list Λ is *Perron* if $\rho(\Lambda) > 0$ and there is a $\lambda \in \Lambda$ of multiplicity one such that $\lambda > |\mu|$ for all other elements $\mu \in \Lambda$. In particular, if Λ is Perron then $\rho(\Lambda) \in \Lambda$.

Given a list Λ, let $f_\Lambda(t) = \Pi_{\lambda \in \Lambda}(t - \lambda)$ denote the monic polynomial whose roots are the elements of Λ, with appropriate multiplicity. For example, if $\Lambda = \langle 1, 1, 2 \rangle$ then $f_\Lambda(t) = (t - 1)^2(t - 2)$. We say that a list Λ is $\mathbb{S}$-*algebraic* if $f_\Lambda(t) \in \mathbb{S}[t]$.

Matrices are assumed to be square. A matrix is called *nonnegative* (respectively, *positive*) if all of its entries are nonnegative (respectively, positive) real numbers. If A is a real matrix, let sp(A) denote the list of (complex) eigenvalues of A and $\mathrm{sp}^\times(A)$ the list of nonzero eigenvalues of A. The spectral radius $\rho(A)$ of A is then just the spectral radius of the list sp(A). We say that A is *Perron* if sp(A) is Perron. Thus a primitive matrix is always Perron.

3. The ♮-construction

Let $P(t) = [p_{ij}(t)]$ be an $r \times r$ matrix over $\mathbb{S}[t]$. We construct a directed graph $\Gamma_{P(t)}$ whose edges are labeled by elements from $\mathbb{S}$. The adjacency matrix of $\Gamma_{P(t)}$ is denoted by $P(t)^\natural$, which has entries in $\mathbb{S}$. The process of passing from $P(t)$ to $P(t)^\natural$ is called the ♮-*construction*.

To describe $\Gamma_{P(t)}$, let $d(j) = \max_{1 \leqslant i \leqslant r} \deg(p_{ij})$. The vertices of $\Gamma_{P(t)}$ are symbols j_k, where $1 \leqslant j \leqslant r$ and $0 \leqslant k \leqslant d(j)$. For $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant d(j)$ put an edge labeled 1 from j_k to j_{k-1}. For each monomial at^k in $p_{ij}(t)$ put an edge labeled a from i_0 to j_k. This completes the construction of $\Gamma_{P(t)}$.

Example 3.1. Let $\mathbb{S} = \mathbb{Z}$ and

$$P(t) = \begin{bmatrix} 2t + 3 & 4t^2 + 5t + 6 \\ 7 & 8t^2 + 9 \end{bmatrix}.$$

The graph $\Gamma_{P(t)}$ is shown in Fig. 1.

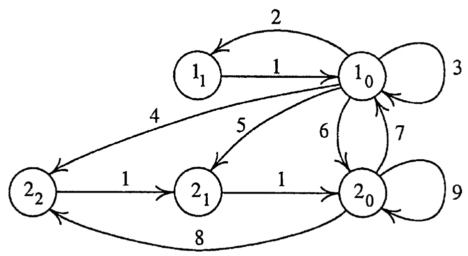

Fig. 1. The graph $\Gamma_{P(t)}$ for Example 3.1.

Using the vertex ordering $1_1, 1_0, 2_2, 2_1, 2_0$, the adjacency matrix of $\Gamma_{P(t)}$ takes the form

$$P(t)^\natural = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 7 & 8 & 0 & 9 \end{bmatrix}.$$

Remark 3.2. (1) If A is a matrix over $\mathbb{S}$, then $A^\natural = A$. Thus every matrix over $\mathbb{S}$ arises from the $\natural$-construction.

(2) The $\natural$-construction can be viewed as a generalization of the companion matrix of a polynomial. For if $P(t) = [p(t)]$ is 1×1 and $m = \deg(p)$, then $P(t)^\natural$ is the companion matrix of $t^m[t - p(t^{-1})]$.

(3) Our construction of $P(t)^\natural$ from $P(t)$ is a variation of the $\sharp$-construction of an $\mathbb{S}$ matrix from $tP(t)$ in [14] (where $\mathbb{S} = Z$). In particular,

$$\det[I - t\{P(t)^\natural\}] = \det[I - t\{tP(t)\}^\natural].$$

The $\natural$-construction generally yields smaller matrices than the $\sharp$-construction, and so is better suited for our purposes.

If A is a matrix over the complex numbers $\mathbb{C}$, then the polynomial

$$\det[I - tA] = \prod_{\lambda \in \mathrm{sp}^\times(A)} (1 - \lambda t)$$

determines the list $\mathrm{sp}^\times(A)$ of nonzero eigenvalues of A. The following result, essentially contained in [3, Theorem 1.7] (see also [4, Section 5.3]), shows that for $A = P(t)^\natural$ this polynomial can be readily computed from the smaller matrix $P(t)$.

Proposition 3.3. *If $P(t)$ is a polynomial matrix over $\mathbb{S}[t]$, then*

$$\det[I - t\{P(t)^\natural\}] = \det[I - tP(t)]. \tag{3.1}$$

Proof. Let $P(t) = [p_{ij}(t)]$ be $r \times r$, and S_r be the permutation group of $\{1, \dots, r\}$. Let $\mathscr{V} = \{j_k : 1 \leqslant j \leqslant r,\ 0 \leqslant k \leqslant d(j)\}$ be the vertex set of $\Gamma_{P(t)}$, and $S(\mathscr{V})$ denote the permutation group of $\mathscr{V}$. Denote the Kronecker function by δ_{ij}.

Consider the expansion of $\det[I - t\{P(t)^\natural\}]$ using permutations in $S(\mathscr{V})$. We first observe that any $\pi \in S(\mathscr{V})$ contributing a nonzero product

$$\prod_{v\in\mathscr{V}} \left[\delta_{v,\pi v} - t\left\{P(t)^\natural{}_{v,\pi v}\right\}\right] \tag{3.2}$$

to this expansion must have a special form. For $1 \leqslant i \leqslant r$ we have that $\pi(i_0) = j_k$ for some $1 \leqslant j \leqslant r$ and $0 \leqslant k \leqslant d(j)$. Observe that for $k \geqslant 1$ the nonzero entries in the j_kth row of $I - t\{P(t)^\natural\}$ can occur only in columns j_k or j_{k-1}. Since $\pi(i_0) = j_k$, we must then have $\pi(j_k) = j_{k-1}$. We then see inductively that $\pi(j_{k-1}) = j_{k-2}, \ldots, \pi(j_1) = j_0$. An analogous argument for predecessors of j_k shows in turn that $\pi(j_{k+1}) = j_{k+1}, \ldots, \pi(j_{d(j)}) = j_{d(j)}$. If a_k denotes the coefficient of t^k in $p_{ij}(t)$, the subproduct of (3.2) over the subset $\{i_0\} \cup \{j_\ell : 1 \leqslant \ell \leqslant d(j)\} \subset \mathscr{V}$ is then $(-1)^k(-a_k t^{k+1})$.

This observation also shows that if $i' \neq i$ and $\pi(i'_0) = j'_{k'}$, then $j' \neq j$. Hence π induces a permutation $\sigma \in S_r$ defined by $\sigma(i) = j$ whenever $\pi(i_0) = j_k$. Clearly π is determined by σ and the choices of k with $0 \leqslant k \leqslant d(j)$. Conversely, each $\sigma \in S_r$ and choice of k's determine a relevant π.

To formalize these observations, define K to be the set of all functions $\kappa : \{1, \ldots, r\} \to \mathbb{Z}_+$ such that $0 \leqslant \kappa(j) \leqslant d(j)$. For each $\sigma \in S_r$ and $\kappa \in K$ define

$$\pi_{\sigma,\kappa}(j_k) = \begin{cases} (\sigma j)_{\kappa(\sigma j)} & \text{for } k = 0, \\ j_{k-1} & \text{for } 1 \leqslant k \leqslant \kappa(j), \\ j_k & \text{for } \kappa(j) < k \leqslant d(j). \end{cases}$$

Let $\mathscr{E}(\sigma) = \{\pi_{\sigma,\kappa} : \kappa \in K\} \subset S(\mathscr{V})$. Clearly the $\mathscr{E}(\sigma)$ are pairwise disjoint for $\sigma \in S_r$. Our previous observations show that $\bigcup_{\in S_r} \mathscr{E}(\sigma)$ contains all permutations in $S(\mathscr{V})$ that could possibly contribute a nonzero term to the expansion of $\det[I - t\{P(t)^\natural\}]$.

Fix $\sigma \in S_r$. The expansion of

$$\prod_{j=1}^{r} \left[\delta_{j,\sigma j} - t p_{j,\sigma j}(t)\right]$$

contains monomials parameterized by K, where $\kappa \in K$ determines which monomial from each polynomial to select to form a product. As observed above, the same monomials appear in the expansion of

$$\sum_{\kappa\in K} (\operatorname{sgn} \pi_{\sigma,\kappa}) \prod_{v\in\mathscr{V}} \left[\delta_{v,\pi_{\sigma,\kappa} v} - t\left\{P(t)^\natural{}_{v,\pi_{\sigma,\kappa} v}\right\}\right],$$

but multiplied by $\prod_{j=1}^r (-1)^{\kappa(j)}$. Since the cycle lengths of $\pi_{\sigma,\kappa}$ increase over those in σ by a total amount $\sum_{j=1}^r \kappa(j)$, it follows that

$$(\operatorname{sgn} \pi_{\sigma,\kappa}) \prod_{j=1}^{r} (-1)^{\kappa(j)} = \operatorname{sgn} \sigma.$$

Hence

$$\sum_{\kappa\in K}(\operatorname{sgn}\pi_{\sigma,\kappa})\prod_{v\in\mathscr{V}}\left[\delta_{v,\pi_{\sigma,\kappa}v}-t\left\{P(t)^{\natural}{}_{v,\pi_{\sigma,\kappa}v}\right\}\right]$$
$$=(\operatorname{sgn}\sigma)\prod_{j=1}^{r}\left[\delta_{j,\sigma j}-tp_{j,\sigma j}(t)\right].$$

Summing over $\sigma\in S_r$ establishes the result. $\square$

Example 3.4. If $P(t)$ is the polynomial matrix in Example 3.1, the reader can verify that

$$\det[I-tP(t)]=\det[I-t\{P(t)^{\natural}\}]=1-12t-17t^2-25t^3-4t^4+16t^5.$$

Remark 3.5. Let Γ be a directed graph. Borrowing terminology from [3], we call a subset R of vertices of Γ a *rome* if Γ has no cycle disjoint from R. Alternatively, R is a rome if every sufficiently long path in Γ must pass through R, so that all roads lead to R. A rome is effectively a cross-section for the path structure of Γ.

For example, if $P(t)$ is an $r\times r$ polynomial matrix, then $\Gamma_{P(t)}$ has a rome $R=\{1_0,2_0,\dots,r_0\}$ of size r. Conversely, suppose that Γ is a directed graph whose edges e are labeled by elements $\operatorname{wt}(e)\in\mathbb{S}$. Suppose that Γ has a rome R of size r. For each ordered pair (i,j) of vertices in R, let Ω_{ij} denote the (finite) set of paths ω from i to j that do not otherwise contain a vertex in R. For each such ω define its length $\ell(\omega)$ to be the number of edges, and its weight to be $\operatorname{wt}(\omega)=\prod_{e\in\omega}\operatorname{wt}(e)\in\mathbb{S}$. Let

$$p_{ij}(t)=\sum_{\omega\in\Omega_{ij}}\operatorname{wt}(\omega)t^{\ell(\omega)-1}\in\mathbb{S}[t],$$

and $P=[p_{ij}(t)]$. If A is the adjacency matrix of Γ, then A and $P(t)^{\natural}$ may be quite different. However, an argument similar to that in Proposition 3.3 shows that $\det[I-tA]=\det[I-t\{P(t)^{\natural}\}]=\det[I-tP(t)]$. Thus our results amount to finding graphs with prescribed spectral behavior having small romes.

4. Manufacturing polynomial matrices

Let A be a $d\times d$ nonsingular matrix over $\mathbb{S}$, and $\mathbb{K}$ be the quotient field of $\mathbb{S}$. It is convenient to use row vectors, and therefore to write the action of matrices on the right. Suppose we have r vectors $\mathbf{x}_1,\dots,\mathbf{x}_r\in\mathbb{S}^d$ whose images under powers of A span $\mathbb{K}^d$. Further suppose that each image $\mathbf{x}_jA$ can be written as an $\mathbb{S}$-linear combination of the $\mathbf{x}_iA^{-k}$ for $1\leqslant i\leqslant r$ and $k\geqslant 0$. Then there are polynomials $p_{ij}(t)\in\mathbb{S}[t]$ such that

$$\begin{aligned} \mathbf{x}_1 A &= \mathbf{x}_1 p_{11}(A^{-1}) + \mathbf{x}_2 p_{12}(A^{-1}) + \cdots + \mathbf{x}_r p_{1r}(A^{-1}), \\ &\vdots \\ \mathbf{x}_r A &= \mathbf{x}_1 p_{r1}(A^{-1}) + \mathbf{x}_2 p_{r2}(A^{-1}) + \cdots + \mathbf{x}_r p_{rr}(A^{-1}). \end{aligned}$$

Let $P(t) = [p_{ij}(t)]$ be the resulting $r \times r$ polynomial matrix. Form $P(t)^\natural$, say of size n. Define a $\mathbb{K}$-linear map $\psi : \mathbb{K}^n \to \mathbb{K}^d$ by $\psi(j_k) = \mathbf{x}_j A^{-k}$. It is routine to check that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{P(t)^\natural} & \mathbb{K}^n \\ \psi\big\downarrow & & \big\downarrow\psi \\ \mathbb{K}^d & \xrightarrow[A]{} & \mathbb{K}^d \end{array}$$

Since the $\mathbf{x}_i$ generate $\mathbb{K}^d$ under powers of A, it follows that ψ is surjective.

This method provides the algebraic machinery to obtain given matrices A as quotients of $\natural$-constructions. The following section shows how to use positivity to control the spectral radius as well as obtain primitivity of $P(t)^\natural$.

5. Small polynomial matrices

In this section we realize a given Perron list as a subset of the spectrum of a primitive nonnegative matrix having the same spectral radius obtained via the $\natural$-constructions from a polynomial matrix that is either 1×1 or 2×2.

Theorem 5.1. *Let Λ be an $\mathbb{S}$-algebraic Perron list of nonzero complex numbers. Then there is a polynomial matrix P(t) over $\mathbb{S}_+[t]$ of size at most two such that $P(t)^\natural$ is primitive, $\rho(\Lambda) = \rho(P(t)^\natural)$, and $\Lambda \subset \mathrm{sp}^\times(P(t)^\natural)$.*

Proof. If $\Lambda = \{\lambda\}$ for some $\lambda \in \mathbb{S}_{++}$, then *P(t)* be the 1×1 constant matrix $[\lambda]$.

Let d denote the cardinality of Λ, which we may now assume is at least 2. Put $\lambda = \rho(\Lambda) \in \Lambda$, $f_\Lambda(t) = \prod_{\mu\in\Lambda}(t-\mu) \in \mathbb{S}[t]$, and let C be the $d \times d$ companion matrix of $f_\Lambda(t)$. If $\mathbf{e}_j$ denotes the jth standard basis vector, then $\mathbf{e}_j C = \mathbf{e}_{j+1}$ for $1 \leqslant j \leqslant d-1$.

Let $\mathbf{v}$ be a left-eigenvector for C corresponding to λ and $V = \mathbb{R}\mathbf{v}$. Denote by W the direct sum of the generalized eigenspaces corresponding to the other elements of Λ, and let π_V denote projection to V along W. Note that $\mathbf{e}_j \notin W$ for $1 \leqslant j \leqslant d$, since W is a C-invariant proper subspace and each $\mathbf{e}_j$ generates $\mathbb{R}^d$ under (positive and negative) powers of C. We identify $\mathbb{R}$ with $\mathbb{R}\mathbf{v}$ via $t \leftrightarrow t\mathbf{v}$, and think of π_V as having range $\mathbb{R}$. Replacing $\mathbf{v}$ with $-\mathbf{v}$ if necessary, we may assume that $\pi_V(\mathbf{e}_1) > 0$, and hence $\pi_V(\mathbf{e}_j) = \pi_V(\mathbf{e}_1 C^{j-1}) = \lambda^{j-1}\pi_V(\mathbf{e}_1) > 0$ for $1 \leqslant j \leqslant d$.

We claim that $\mathbf{v}, \mathbf{v}-\mathbf{e}_1, \ldots, \mathbf{v}-\mathbf{e}_{d-1}$ are linearly independent. For if not, then $\mathbf{v}$ would be a linear combination $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_{d-1}\mathbf{e}_{d-1}$. Taking dth coordinates of $\mathbf{v}C = \lambda\mathbf{v}$ shows that $v_{d-1} = 0$, and so on, contradicting $\mathbf{v} \neq 0$ and proving our claim. Hence the $\mathbb{R}_+$-cone generated by $\mathbf{v}, \mathbf{v} - \mathbf{e}_1, \dots, \mathbf{v} - \mathbf{e}_{d-1}$ has nonempty interior. This interior must therefore contain some $\mathbf{u} \in \mathbb{S}^d$ of the form

$$\mathbf{u} = c_0\mathbf{v} + c_1(\mathbf{v} - \mathbf{e}_1) + \cdots + c_{d-1}(\mathbf{v} - \mathbf{e}_{d-1}),$$

where $c_j > 0$ for $0 \leqslant j \leqslant d-1$ and in addition $\pi_V(\mathbf{u}) > 0$. Thus

$$\mathbf{v} = \frac{1}{c_0 + c_1 + \cdots + c_{d-1}}(\mathbf{u} + c_1\mathbf{e}_1 + \cdots + c_{d-1}\mathbf{e}_{d-1})$$

lies in the interior of the $\mathbb{R}_+$-cone K generated by $\mathbf{e}_1, \dots, \mathbf{e}_{d-1}$ and $\mathbf{u}$, and in addition $K \cap W = \{0\}$.

Our goal is to show that for all sufficiently large N there are elements a_j, b_j, a, and b in $\mathbb{S}_{++}$ such that

$$\begin{aligned}\mathbf{e}_d C^N &= a_1\mathbf{e}_1 + \cdots + a_{d-1}\mathbf{e}_{d-1} + a_d\mathbf{e}_d + a\mathbf{u}\\ &= a_1\mathbf{e}_d C^{-d+1} + \cdots + a_{d-1}\mathbf{e}_d C^{-1} + a_d\mathbf{e}_d + a\mathbf{u},\\ \mathbf{u}C^N &= b_1\mathbf{e}_1 + \cdots + b_{d-1}\mathbf{e}_{d-1} + b_d\mathbf{e}_d + b\mathbf{u}\\ &= b_1\mathbf{e}_d C^{-d+1} + \cdots + b_{d-1}\mathbf{e}_d C^{-1} + b_d\mathbf{e}_d + b\mathbf{u},\end{aligned} \tag{*}$$

Suppose for now this goal has been met. Then applying C^{-N+1} to both equations puts us into the situation described in Section 4, with $r = 2$, $\mathbf{x}_1 = \mathbf{e}_d$, $\mathbf{x}_2 = \mathbf{u}$, and

$$P(t) = \begin{bmatrix} a_1t^{N+d-2} + a_2t^{N+d-3} + \cdots + a_{d-1}t^N + a_dt^{N-1} & at^{N-1}\\ b_1t^{N+d-2} + b_2t^{N+d-3} + \cdots + b_{d-1}t^N + b_dt^{N-1} & bt^{N-1}\end{bmatrix}.$$

The graph $\Gamma_{P(t)}$ is strongly connected because $a_j, b_j, a, b > 0$. It also has period one since $d \geqslant 2$ and $\gcd(N-1, N) = 1$. Therefore $P(t)^\natural$ is primitive. The map ψ defined in Section 4 shows that C is a quotient of $P(t)^\natural$, so that $\Lambda = \mathrm{sp}(C) \subset \mathrm{sp}^\times(P(t)^\natural)$, and hence $\rho(\Lambda) \leqslant \rho(P(t)^\natural)$. The Perron eigenvector for $P(t)^\natural$ is mapped by ψ to a vector which is nonzero (it is a strictly positive combination of $\mathbf{e}_1, \dots, \mathbf{e}_{d-1}$, and $\mathbf{u}$) and which is therefore an eigenvector of C with eigenvalue $\rho(P(t)^\natural)$, proving that $\rho(\Lambda) \geqslant \rho(P(t)^\natural)$. This completes the proof except for establishing $(*)$.

To prove that $(*)$ holds for sufficiently large N, we consider separately the cases $\mathbb{S} = \mathbb{Z}$ and $\mathbb{S}$ dense in $\mathbb{R}$.

First suppose that $\mathbb{S} = \mathbb{Z}$. Since Λ is $\mathbb{Z}$-algebraic and $|\Lambda| = d \geqslant 2$, it follows that $|\prod_{\mu\in\Lambda}\mu| = |f_\Lambda(0)| \geqslant 1$, and hence $\lambda = \rho(\Lambda) > 1$. Let $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}\mathbf{e}_{d-1} \oplus \mathbb{Z}\mathbf{u}$ be the lattice generated by $\mathbf{e}_1, \dots, \mathbf{e}_{d-1}, \mathbf{u}$. Choose M large enough so that every translate of $Q = [1, M]^d$ contains an element of L. Suppose that $\mathbf{w} \in \mathbb{Z}^d$ has the property that $\mathbf{w} - Q$ is contained in the interior K° of the cone K. Then $\mathbf{w} - Q$ contains an element $\mathbf{x} = \mathbf{w} - \mathbf{q}$ in L, say $\mathbf{x} = n_1\mathbf{e}_1 + \cdots + n_{d-1}\mathbf{e}_{d-1} + n\mathbf{u}$ with n_j, $n \in \mathbb{Z}$. These coefficients n_j, n must then be in $\mathbb{Z}_{++}$ because $x \in K^\circ$ and the representation of x as a linear combination of the linearly independent vectors $e_1, \ldots, e_{d-1}, u$ is unique. Now $\mathbf{q} = \mathbf{w} - \mathbf{x} \in \mathbb{Z}^d \cap Q$, and so $\mathbf{q} = q_1\mathbf{e}_1 + \cdots + q_d\mathbf{e}_d$ with all $q_j \in \mathbb{Z}_{++}$. Thus

$$\mathbf{w} = \mathbf{x} + \mathbf{q} = (n_1 + q_1)\mathbf{e}_1 + \cdots + (n_{d-1} + q_{d-1})\mathbf{e}_{d-1} + q_d\mathbf{e}_d + n\mathbf{u},$$

where the coefficient of each vector lies in $\mathbb{Z}_{++}$. Since $\mathbf{v}$ is the dominant eigendirection, its eigenvalue $\lambda > 1$, and $\pi_V(\mathbf{e}_d) > 0, \pi_V(\mathbf{u}) > 0$, it follows that for all sufficiently large N both $\mathbf{e}_d C^N - Q$ and $\mathbf{u}\, C^N - Q$ are contained in K°. By what we have just done, this shows that $(*)$ is valid in the case $\mathbb{S} = \mathbb{Z}$.

Finally, suppose that $\mathbb{S}$ is dense in $\mathbb{R}$. Let $K_\mathbb{S}$ denote the set of all elements in K of the form $s_1\mathbf{e}_1 + \cdots + s_{d-1}\mathbf{e}_{d-1} + s\mathbf{u}$, where $s_j, s \in \mathbb{S}_{++}$. Clearly $K_\mathbb{S}$ is dense in K. Let $\mathbf{w}$ denote any vector in $\mathbb{S}^d$ lying in the interior K° of K. Then $(\mathbf{w} - (0,1)^d) \cap K^\circ$ is open and nonempty, and so contains some vector $\mathbf{x} = \mathbf{w} - \mathbf{q} \in K_\mathbb{S} \subset \mathbb{S}^d$. By definition, $\mathbf{x}$ has the form

$$\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_{d-1}\mathbf{e}_{d-1} + x\mathbf{u},$$

where $x_j, x \in \mathbb{S}_{++}$. Then $\mathbf{q} = \mathbf{w} - \mathbf{x} \in \mathbb{S}^d \cap (0,1)^d$, so that $\mathbf{q} = q_1\mathbf{e}_1 + \cdots + q_d\mathbf{e}_d$, where $q_j \in \mathbb{S}_{++}$. Hence

$$\mathbf{w} = \mathbf{x} + \mathbf{q} = (x_1 + q_1)\mathbf{e}_1 + \cdots + (x_{d-1} + q_{d-1})\mathbf{e}_{d-1} + q_d\mathbf{e}_d + x\mathbf{u},$$

where each coefficient lies in $\mathbb{S}_{++}$. Since $\mathbf{v}$ is the dominant eigendirection and $\pi_V(\mathbf{e}_d) > 0, \pi_V(\mathbf{u}) > 0$, both $\mathbf{e}_d C^N$ and $\mathbf{u}C^N$ are in K° for all sufficiently large N. By the above, we have established $(*)$ when $\mathbb{S}$ is dense, and completed the proof. □

6. Examples and remarks

We illustrate how the ideas in the proof of Theorem 5.1 work in three concrete situations, and also make some general remarks.

Example 6.1. Let $\mathbb{S} = \mathbb{Z}$ and $\Lambda = \langle 2, 1\rangle$. Then Λ is an $\mathbb{Z}$-algebraic Perron list with $\lambda = \rho(\Lambda) = 2$. Using the notation from the proof of Theorem 5.1,

$$C = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{v} = [-1 \quad 1], \quad \text{and} \quad W = \mathbb{R} \cdot [-2 \quad 1].$$

We pick $\mathbf{u} = \mathbf{v} + (\mathbf{v} - \mathbf{e_1}) = [-3\ \ 2]$, so that $\pi_V(\mathbf{u}) > 0$ and $\mathbf{v}$ is in the interior K° of the cone K generated by $\mathbf{e}_1$ and $\mathbf{u}$. Here $L = \mathbb{Z}\mathbf{e}_1 + 2\mathbb{Z}\mathbf{e}_2$, so we can let $Q = [1,2]^2$. The minimal N for which both $\mathbf{e}_2 C^N - Q$ and $\mathbf{u}\, C^N - Q$ are contained in K° turns out to be $N = 4$. We compute

$$\mathbf{e}_2 C^4 - [1 \quad 1] = [-31 \quad 30] = 14[1 \quad 0] + 15[-3 \quad 2] \in L \quad \text{and}$$

$$\mathbf{u}C^4 - [1 \quad 1] = [-19 \quad 16] = 5[1 \quad 0] + 8[-3 \quad 2] \in L.$$

Continuing with the method of the proof, we have

$$\mathbf{e}_2C^4 = (14\mathbf{e}_1 + 15\mathbf{u}) + (\mathbf{e}_1 + \mathbf{e}_2) = 15\mathbf{e}_1 + \mathbf{e}_2 + 15\mathbf{u},$$

$$\mathbf{u}C^4 = (5\mathbf{e}_1 + 8\mathbf{u}) + (\mathbf{e}_1 + \mathbf{e}_2) = 6\mathbf{e}_1 + \mathbf{e}_2 + 8\mathbf{u}.$$

Hence

$$\mathbf{e}_2C = 15\mathbf{e}_1C^{-3} + \mathbf{e}_2C^{-3} + 15\mathbf{u}C^{-3} = \mathbf{e}_2(15C^{-4} + C^{-3}) + \mathbf{u}(15C^{-3}),$$

$$\mathbf{u}C = 6\mathbf{e}_1C^{-3} + \mathbf{e}_2C^{-3} + 8\mathbf{u}C^{-3} = \mathbf{e}_2(6C^{-4} + C^{-3}) + \mathbf{u}(8C^{-3}).$$

From this we obtain

$$P(t) = \begin{bmatrix} 15t^4 + t^3 & 15t^3 \\ 6t^4 + t^3 & 8t^3 \end{bmatrix}.$$

Then $P(t)^\natural$ is a 9×9 primitive integral matrix whose characteristic polynomial is

$$t^9 - 9t^5 - 15t^4 - 7t + 30 = (t-2)(t-1)f(t),$$

where $f(t)$ is an irreducible polynomial of degree 7, all of whose roots have absolute value between 1.46 and 1.86. Thus $P(t)$ satisfies our requirements.

Example 6.2. Again let $\mathbb{S} = \mathbb{Z}$ and put $g(t) = t^3 + 3t^2 - 15t - 46$. Denote the roots of $g(t)$ by $\lambda \cong 3.89167$, $\mu_1 \cong -3.21417$, and $\mu_2 \cong -3.67750$. Then $\Lambda = \langle \lambda, \mu_1, \mu_2 \rangle$ is a $\mathbb{Z}$-algebraic Perron list. The companion matrix C of $g(t)$ turns out to have a positive left-eigenvector $\mathbf{v}$ corresponding to λ. Thus we can let $\mathbf{u} = \mathbf{e_3}$ since $\mathbf{v}$ lies in the interior of the positive orthant $K = \mathbb{R}^3_+$. Hence we can use the manufacturing technique in Section 4 with $r = 1$ and the single vector $\mathbf{x_1} = \mathbf{e_1}$, yielding a 1×1 polynomial matrix. However, since μ_1 and μ_2 are negative and close in size to λ, it takes a large value of N to force $\mathbf{e}_1C^N$ inside K. By direct computation we find the smallest N which works is $N = 49$ and that $\mathbf{e}_1C^{49} = [a \;\; b \;\; c]$, where

$$a = 36488554855989658309872537378,$$
$$b = 11571239128278403776343659967,$$
$$c = 674104003853663694665556470.$$

Hence

$$\mathbf{e}_1C = a\mathbf{e}_1C^{-48} + b\mathbf{e}_1C^{-47} + c\mathbf{e}_1C^{-46},$$

resulting in $p(t) = at^{48} + bt^{47} + ct^{46}$. Then $[p(t)]^\natural$ is a 49×49 primitive integral matrix whose characteristic polynomial is $g(t)h(t)$, where $h(t)$ is an irreducible polynomial of degree 46 all of whose roots have absolute value between 3.709 and $3.8915 < \lambda$ and the bounds are optimal to the given accuracy.

Example 6.3. For this example we use the dense unital subring $\mathbb{S} = \mathbb{Z}[1/6]$. Let $\Lambda = \langle 1/2, 1/3 \rangle$, an $\mathbb{S}$-algebraic Perron list. Here

$$C = \begin{bmatrix} 0 & 1 \\ -1/6 & 5/6 \end{bmatrix}, \quad \mathbf{v} = [-1 \quad 3], \quad \text{and} \quad W = \mathbb{R} \cdot [-1 \ \ 2].$$

We pick $\mathbf{u} = [-2 \ \ 5]$, and let K be the $\mathbb{R}_+$-cone generated by $\mathbf{e}_1$ and $\mathbf{u}$.

First notice that although

$$\mathbf{u}C = [-5/6 \quad 13/6] \in K^\circ \cap \mathbb{S}^2$$

has coordinates in $\mathbb{S}$ and is an $\mathbb{R}_{++}$-combination of $\mathbf{e}_1$ and $\mathbf{u}$, it is *not* an $\mathbb{S}_{++}$-combination of $\mathbf{e}_1$ and $\mathbf{u}$, since

$$\mathbf{u}C = \frac{1}{30}\mathbf{e}_1 + \frac{13}{30}\mathbf{u}$$

is the unique representation of $\mathbf{u}C$ as a linear combination of $\mathbf{e}_1$ and $\mathbf{u}$, and $1/30 \notin \mathbb{S}$. This difficulty explains the necessity in our proof of getting $\mathbb{S}_{++}$ combinations close to the given vectors.

Here both $\mathbf{e}_2C$ and $\mathbf{u}C$ are in K°. We need to find vectors $a\mathbf{e}_1 + b\mathbf{u}$ that are close to the given vectors, which is effectively a problem in Diophantine approximation of rationals by elements of $\mathbb{S}$.

For $\mathbf{e}_2C$, we seek $a, b \in \mathbb{S}_{++}$ so that $\mathbf{x} = a\mathbf{e}_1 + b\mathbf{u} = [a - 2b \ \ 5b]$ is coordinate-wise less than but close to $\mathbf{e}_2C = [-1/6 \ \ 5/6]$. Thus $b < 1/6$, so we pick $b = 5/36$. Then $a < -1/6 + 10/36 = 4/36$ and we pick $a = 3/36 = 1/12$. Then

$$\mathbf{e}_2C - \frac{1}{12}\mathbf{e}_1 - \frac{5}{36}\mathbf{u} = \frac{1}{36}\mathbf{e}_1 + \frac{5}{36}\mathbf{e}_2,$$

so that

$$\mathbf{e}_2C = \mathbf{e}_2\left(\frac{1}{9}C^{-1} + \frac{5}{36}\right) + \mathbf{u}\left(\frac{5}{36}\right).$$

A similar calculation gives

$$\mathbf{u}C = \mathbf{e}_2\left(\frac{1}{36}C^{-1} + \frac{1}{72}\right) + \mathbf{u}\left(\frac{93}{216}\right).$$

Hence we find

$$P(t) = \begin{bmatrix} \frac{1}{9}t + \frac{5}{36} & \frac{5}{36} \\ \frac{1}{36}t + \frac{1}{72} & \frac{93}{216} \end{bmatrix}.$$

Then $P(t)^\natural$ is a 3×3 primitive matrix over $\mathbb{S}_+$ whose eigenvalues are $1/2,\ 1/3$, and $-19/72$.

Remark 6.4. The singleton case $\Lambda = \langle\lambda\rangle$ in Theorem 5.1 was handled using a 1×1 matrix. With the single exception of the case $\mathbb{S} = \mathbb{Z}$ and $\Lambda = \langle 1 \rangle$, a 2×2

polynomial matrix can also be found satisfying the desired conclusions. For if $\lambda > 1$ apply the proof to $\langle\lambda, 1\rangle$, and if $\lambda < 1$ apply it to $\langle\lambda, \lambda^2\rangle$. If $\lambda = 1$ and $\mathbb{S}$ is dense, pick $\mu \in \mathbb{S} \cap (0, 1)$ and apply the proof to $\langle 1, \mu\rangle$.

To discuss the exceptional case, suppose that A is an $r \times r$ primitive integral matrix, where $r \geqslant 2$. Then $A^n > 0$ for some $n \geqslant 1$. The spectral radius of A^n is bounded below by the minimum of the row sums of A^n, and hence by r. Thus $\rho(A) = \rho(A^n)^{1/n} \geqslant r^{1/n} > 1$. This shows that when $\mathbb{S} = \mathbb{Z}$ and $\Lambda = \langle 1\rangle$ there cannot be a 2×2 polynomial matrix satisfying the conclusions of Theorem 5.1.

Remark 6.5. The construction in the proof of Theorem 5.1 typically introduces additional nonzero spectrum. When $\mathbb{S} = \mathbb{Z}$ there is a further restriction on a $\mathbb{Z}$-algebraic Perron list Λ that it be exactly the nonzero spectrum of a primitive integral matrix. Define $\operatorname{tr}(\Lambda^n) = \sum_{\lambda\in\Lambda} \lambda^n$, and the nth net trace to be

$$\operatorname{tr}_n(\Lambda) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \operatorname{tr}(\Lambda^d),$$

where μ is the Möbius function. If there were a primitive integral matrix A with $\operatorname{sp}^\times(A) = \Lambda$, then $\operatorname{tr}_n(\Lambda)$ would count the number of orbits of least period n in an associated dynamical system (see [19, p. 348]). Hence a necessary (and easily checked) condition for there to be a primitive integral matrix A such that $\operatorname{sp}^\times(A) = \Lambda$ is that $\operatorname{tr}_n(\Lambda) \geqslant 0$ for all $n \geqslant 1$. Kim et al. [13] have shown that this condition also suffices. Their remarkable proof uses, among other things, polynomial matrices to find the required A.

When $\mathbb{S} \neq \mathbb{Z}$, an obviously necessary condition replaces the net trace condition above: if $\operatorname{tr}_n(\Lambda) > 0$ then $\operatorname{tr}_{kn}(\Lambda) > 0$ for all $k \geqslant 1$. The Spectral conjecture in [6] states that when $\mathbb{S} \neq \mathbb{Z}$ this condition is sufficient for an $\mathbb{S}$-algebraic Perron list to be the nonzero spectrum of a primitive matrix over $\mathbb{S}_+$. The Spectral Conjecture was proven in [6] for the case $\mathbb{S} = \mathbb{R}$, and some other cases.

Remark 6.6. There are constraints of Johnson–Loewy–London type [11,20] which put lower bounds on the size of a polynomial matrix $P(t)$ for which $P(t)^\natural$ realizes a given Perron list Λ. For example, for $\mathbb{S} = \mathbb{Z}$, if $\operatorname{tr}_1(\Lambda) = n$ and $\rho(\Lambda) < 2$, then the size of $P(t)$ must be at least n (otherwise a diagonal entry of $P(t)$ would have a constant term 2 or greater, forcing $\rho(\Lambda) \geqslant 2$). Without trying here to formulate these constraints carefully, it seems reasonable to us to expect that they may give nearly sharp bounds on the smallest size of a polynomial matrix realizing a given nonzero spectrum.

Remark 6.7. As pointed out in [4], one consequence of work by Perrin [22] is a version of Theorem 5.1 without the additional property that $P(t)^\natural$ is primitive. This property is significant because applications of nonnegative matrices are often reduced to or based on the primitive case.

Remark 6.8. The technique in Section 4 of manufacturing nonnegative matrices using a general matrix with Perron spectrum was introduced in [17] and used subsequently in various guises (e.g. [8, Theorem 5.14] and [10,18]).

7. Handelman's theorem

We use the geometric point of view developed above to recover the main parts of Handelman's result [9, Theorem 5].

Suppose that $P(t) = [p(t)]$ with $p(t) \in \mathbb{S}_+[t]$. By Proposition 3.3, every nonzero eigenvalue μ of $P(t)^\natural$ satisfies $1 = \mu^{-1}p(\mu^{-1})$. Several people have observed that strict monotonicity of $tp(t)$ for $t > 0$ then implies that $\mathrm{sp}^\times(P(t)^\natural)$ cannot have any positive members except for the spectral radius $\rho(P(t)^\natural)$. The following result of Handelman provides a converse to this, and is relevant, for example, in determining the possible entropies of uniquely decipherable codes [10]. Handelman's original proof employed results about the coefficients of large powers of polynomials.

Our proof combines ideas from the previous section with the following elementary property of linear transformations. In order to state this property, recall that the *nonnegative cone* generated by a set of vectors in a real vector space is the collection of all finite nonnegative linear combinations of vectors in the set.

Lemma 7.1. *Let B be an invertible linear transformation of a finite-dimensional real vector space and suppose that B has no positive eigenvalue. Then for every vector* $\mathbf{e}$, *the nonnegative cone generated by* $\{\mathbf{e}B^m : m \geqslant 0\}$ *is a vector subspace.*

Proof. Given a vector $\mathbf{e}$, let K be the nonnegative cone generated by the $\{\mathbf{e}B^m : m \geqslant 0\}$, and let W be the real vector space generated by $\{\mathbf{e}B^m : m \geqslant 0\}$. We claim that $K = W$.

For suppose that $K \neq W$. Let $\overline{K}$ denote the closure of K. Since proper cones are contained in half-spaces [23, Theorem 11.5], it follows that $\overline{K} \neq W$. Then $U = \overline{K} \cap (-\overline{K})$ is a subspace of W such that $U \subsetneq \overline{K}$. Both W and U are mapped into themselves by B. Hence the quotient map D of B on W/U maps the closed cone $\overline{K}/U$ into itself. Furthermore, $\overline{K}/U$ has nonempty interior and $(\overline{K}/U) \cap (-\overline{K}/U) = \{\mathbf{0}\}$. It then follows (see [1] or [2, p. 6]) that the spectral radius λ_D of D is an eigenvalue of D. Because B is invertible and W/U is nonzero. we have that $\lambda_D > 0$. But every eigenvalue of D is an also eigenvalue of B, contradicting the hypothesis on B. $\square$

Theorem 7.2. *Let Λ be an $\mathbb{S}$-algebraic Perron list of nonzero complex numbers having no other positive elements except its spectral radius. Then there is a* 1×1 *polynomial matrix $P(t)$ over $\mathbb{S}_+[t]$ such that $P(t)^\natural$ is primitive*, $\rho(\Lambda) = \rho(P(t)^\natural)$, *and* $\Lambda \subset \mathrm{sp}^\times(P(t)^\natural)$.

Proof. We use the same notation as in the proof of Theorem 5.1, except we do not need the auxiliary vector $\mathbf{u}$. As in that proof, d is the cardinality of Λ, $V = \mathbb{R}\mathbf{v}$

is the dominant eigendirection for the companion matrix C of $f_\Lambda(t)=\prod_{\mu\in\Lambda}(t-\mu)\in\mathbb{S}[t]$, and W is the complementary C-invariant subspace. Here the case $d=1$ is trivial, so we assume that $d\geqslant 2$.

Let B be the restriction of C to W, and $\mathbf{e}$ be the projection of $\mathbf{e}_1$ to W along V. The form of the companion matrix shows that $\{\mathbf{e}B^m : m\geqslant 0\}$ generates the vector space W. It then follows from Lemma 7.1 that $\mathbf{0}$ is in the strict interior of the convex hull of a finite number of the $\mathbf{e}B^m$. Thus there is an $M\geqslant d$ such that $\mathbf{v}$ is contained in the strict interior of the positive cone H generated by $\{\mathbf{e}_1C^m : 0\leqslant m\leqslant M\}$. Let I denote the set of nonnegative integral combinations of the $\{\mathbf{e}_1C^m : 0\leqslant m\leqslant M\}$. It is routine to show that I is syndetic in H, so that there is an $a>0$ such that if $\mathbf{x}-[1,a]^d\subset H$ then $(\mathbf{x}-[1,a]^d)\cap I\neq\emptyset$.

Since $\mathbf{v}$ is the dominant eigendirection and $\pi_V(\mathbf{e}_1)>0$, it follows that for all sufficiently large $N>M$ we have that $\mathbf{e}_1C^N-[1,a]^d\subset H$. Hence there are $v_j\in[1,a]$ and $w_m\in\mathbb{Z}_+\subset\mathbb{S}_+$ such that

$$\mathbf{e}_1C^N-\sum_{j=1}^{d}v_j\mathbf{e}_j=\sum_{m=0}^{M}w_m\mathbf{e}_1C^m.$$

Since $\mathbf{e}_1C^m\in\mathbb{S}^d$ for all $m\geqslant 0$, we see that each $v_j\in\mathbb{S}\cap[1,a]\subset\mathbb{S}_{++}$. Applying C^{-N+1} then shows that

$$\mathbf{e}_1C=\sum_{j=1}^{d}v_j\mathbf{e}_1C^{-N+j}+\sum_{m=0}^{M}w_m\mathbf{e}_1C^{-N+m+1}.$$

Thus we are again in the situation of Section 4, with $r=1$ and $\mathbf{x}_1=\mathbf{e}_1$. Let $P=\left[p(t)\right]$ be the resulting 1×1 matrix over $\mathbb{S}_+[t]$. Since $v_j>0$ for $1\leqslant j\leqslant d$ and $d\geqslant 2$, it follows that $P(t)^\natural$ is primitive. The same arguments as before now show that $\rho(\Lambda)=\rho(P(t)^\natural)$ and $\Lambda\subset\mathrm{sp}^\times(P(t)^\natural)$. $\square$

8. Direct limit modules

A matrix A over $\mathbb{S}$ induces an automorphism $\widehat{A}$ of its associated direct limit $\mathbb{S}$-module $G_\mathbb{S}(A)$ (the definitions are given below). The isomorphism class of the $\mathbb{S}$-module automorphism $\widehat{A}$ determines the nonzero spectrum of A, and often gives finer information. In the case $\mathbb{S}$ is a field, $\widehat{A}$ is the linear transformation obtained by restricting A to the maximal subspace on which it acts nonsingularly, and such an $\widehat{A}$ is classified by its rational canonical form. For more complicated $\mathbb{S}$, the classification of $\widehat{A}$ is more subtle (see [7] and its references): the isomorphism class of $\widehat{A}$ is determined by and determines the shift equivalence class over $\mathbb{S}$ of the matrix A (the "algebraic shift equivalence" class in [7]), which in the case $\mathbb{S}=\mathbb{Z}$ is an important invariant for symbolic dynamics [19].

Let $\mathbb{S}[t^\pm]$ denote the ring $\mathbb{S}[t,t^{-1}]$ of Laurent polynomials with coefficients in $\mathbb{S}$. As we work with polynomial matrices, it will be convenient for us to consider $G_\mathbb{S}(A)$

as an $\mathbb{S}[t^{\pm}]$-module, by letting t^{-1} act by $\widehat{A}$ (the convention of using t^{-1} here rather than t will be explained later). Knowing the class of $G_{\mathbb{S}}(A)$ as an $\mathbb{S}[t^{\pm}]$-module is equivalent to knowing the class of $\widehat{A}$ as an $\mathbb{S}$-module automorphism. We let $g_{\mathbb{S}}(A)$ denote the cardinality of the smallest set of generators of the $\mathbb{S}[t^{\pm}]$-module $G_{\mathbb{S}}(A)$.

Our main result of this section sharpens Theorem 5.1 to show that if A is Perron, then we can always find a $P(t)$ over $\mathbb{S}_+[t]$ of size at most $g_{\mathbb{S}}(A)+1$ so that $P(t)^{\natural}$ is primitive with the same spectral radius as A and there is an $\mathbb{S}[t^{\pm}]$-module epimorphism $G_{\mathbb{S}}(P(t)^{\natural}) \to G_{\mathbb{S}}(A)$. This result implies Theorem 5.1 by letting A be the companion matrix of $f_A(t)$. We will also see that the size of $P(t)$ here must always be at least $g_{\mathbb{S}}(A)$, and for some A must be at least $g_{\mathbb{S}}(A)+1$.

Now we turn to the promised definitions. We first recall the definition of direct limits, using the directed set $(\mathbb{Z}, \leqslant)$, of systems of modules over a commutative ring R. For every $i \in \mathbb{Z}$ let M_i be an R-module, and for all $i \leqslant j$ let $\phi_{ij} : M_i \to M_j$ be an R-homomorphism such that ϕ_{ii} is the identity on M_i, and if $i \leqslant j \leqslant k$ then $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$. Then $(\{M_i\}, \{\phi_{ij}\})$ is called a *directed system* of R-modules. The *direct limit* of such a system is the R-module

$$(\oplus_{i\in\mathbb{Z}} M_i)/N,$$

where N is the R-submodule of the direct sum generated by elements of the form

$$\big(\ldots, 0, a_i, 0, \ldots, 0, -\phi_{ij}(a_i), 0, \ldots\big), \tag{8.1}$$

where $a_i \in M_i$ occurs in the ith coordinate and $-\phi_{ij}(a_i) \in M_j$ in the jth coordinate.

To specialize to our situation, let A be a $d \times d$ matrix over $\mathbb{S}$. Consider the directed system $(\{M_i\}, \{\phi_{ij}\})$ of $\mathbb{S}$-modules, where $M_i = \mathbb{S}^d$ for all $i \in \mathbb{Z}$ and $\phi_{ij} = A^{j-i}$ for $i \leqslant j$. The direct limit of this system is called the *direct limit* $\mathbb{S}$*-module* of A, and is denoted by $G_{\mathbb{S}}(A)$. Thus a typical element of $G_{\mathbb{S}}(A)$ has the form $(\mathbf{s}_i) + N$, where $(\mathbf{s}_i) \in \mathbb{S}^d$ for all i and $\mathbf{s}_i = \mathbf{0}$ for almost all i. Using members of N of the form (8.1), each element $(\mathbf{s}_i) \in \bigoplus_{\mathbb{Z}} \mathbb{S}^d$ is equivalent modulo N to one of the form $(\ldots, \mathbf{0}, \mathbf{0}, \mathbf{s}, \mathbf{0}, \mathbf{0}, \ldots)$ with at most one nonzero entry.

The $\mathbb{S}$-module homomorphism $\widehat{A}$ of $G_{\mathbb{S}}(A)$ is defined by $\widehat{A} : (\mathbf{s}_i) + N \mapsto (\mathbf{s}_i A) + N$. To see that $\widehat{A}$ is an automorphism note that $(\mathbf{s}_i A) + N = (\mathbf{s}_{i+1}) + N$, so $\widehat{A}$ agrees with the automorphism of $G_{\mathbb{S}}(A)$ induced by the left-shift on the direct sum.

There is a more concrete description of the direct limit $\mathbb{S}$-module. To describe this, recall that $\mathbb{K}$ denotes the quotient field of $\mathbb{S}$. Define the *eventual range* of A to be

$$\mathscr{R}(A) = \bigcap_{j=1}^{\infty} \mathbb{R}^d A^j = \bigcap_{j=1}^{d} \mathbb{R}^d A^j.$$

Then the restriction $A^{\times}$ of A to $\mathscr{R}(A)$ is an invertible linear transformation. Set

$$\widetilde{G}_{\mathbb{S}}(A) = \big\{\mathbf{x} \in \mathscr{R}(A) \cap \mathbb{K}^d : \mathbf{x}A^m \in \mathbb{S}^d \text{ for some } m \geqslant 0\big\}.$$

The restriction $\widetilde{A}$ of A to $\widetilde{G}_{\mathbb{S}}(A)$ is an $\mathbb{S}$-module automorphism of $\widetilde{G}_{\mathbb{S}}(A)$.

Lemma 8.1. *There is an $\mathbb{S}$-module isomorphism between $G_{\mathbb{S}}(A)$ and $\widetilde{G}_{\mathbb{S}}(A)$ which intertwines $\widehat{A}$ and $\widetilde{A}$.*

Proof. As observed above, each element $(\mathbf{s}_i) + N \in G_{\mathbb{S}}(A)$ has a representation as $(\dots, \mathbf{0}, \mathbf{0}, \mathbf{s}_i, \mathbf{0}, \mathbf{0}, \dots) + N$, where $\mathbf{s}_i$ occurs in the ith coordinate. By using another element of N of the form (8.1) and increasing i if necessary, we may also assume that $\mathbf{s}_i \in \mathscr{R}(A) \cap \mathbb{K}^d$. Define $\psi : G_{\mathbb{S}}(A) \to \widetilde{G}_{\mathbb{S}}(A)$ by mapping such an element $(A^\times)^{-i}\mathbf{s}_i \in \widetilde{G}_{\mathbb{S}}(A)$. It is routine to show that ψ is a well-defined isomorphism which intertwines $\widehat{A}$ and $\widetilde{A}$. $\square$

In view of this result, we will often identify $G_{\mathbb{S}}(A)$ with $\widetilde{G}_{\mathbb{S}}(A)$.

Example 8.2. (a) Let $d = 1$, $\mathbb{S} = \mathbb{Z}$, and $A = [2]$. Then $\widetilde{G}_{\mathbb{S}}(A) = \widetilde{G}_{\mathbb{Z}}([2]) = \mathbb{Z}[1/2]$, and $\widetilde{A}$ acts by multiplication by 2.

(b) Let $d = 2$, $\mathbb{S} = \mathbb{Z}$,

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $\widetilde{G}_{\mathbb{Z}}(B) = \mathbb{Z}[1/2] \cdot [1, 1]$, and $\widetilde{B}$ again acts by multiplication by 2.

Here A and B give isomorphic direct limit $\mathbb{S}[t^{\pm}]$-modules.

Remark 8.3. Since $A^\times$ is invertible over $\mathbb{S}[1/(\det A^\times)]$, it follows that

$$\mathscr{R}(A) \cap \mathbb{S}^d \subseteq G_{\mathbb{S}}(A) \subseteq \mathscr{R}(A) \cap \mathbb{S}[1/(\det A^\times)]^d.$$

Hence if $1/(\det A^\times) \in \mathbb{S}$, then $G_{\mathbb{S}}(A) = \mathscr{R}(A) \cap \mathbb{S}^d$, and in particular $G_{\mathbb{K}}(A) = \mathscr{R}(A) \cap \mathbb{K}^d$.

Notice that $I - tA : \mathbb{S}[t^{\pm}]^d \to \mathbb{S}[t^{\pm}]^d$ is an $\mathbb{S}[t^{\pm}]$-module homomorphism. Denote its cokernel $\mathbb{S}[t^{\pm}]$-module by

$$\operatorname{coker}(I - tA) = \mathbb{S}[t^{\pm}]^d \big/ \mathbb{S}[t^{\pm}]^d (I - tA).$$

Lemma 8.4. *Let A be a matrix over $\mathbb{S}$. Then there is an $\mathbb{S}[t^{\pm}]^d$-module isomorphism between $G_{\mathbb{S}}(A)$ and* $\operatorname{coker}(I - tA)$.

Proof. There are obvious $\mathbb{S}$-module identifications

$$\oplus_{\mathbb{Z}} \mathbb{S}^d \cong \oplus_{i \in \mathbb{Z}} \mathbb{S}^d t^i \cong \mathbb{S}[t^{\pm}]^d.$$

In the definition of $G_{\mathbb{S}}(A)$, the $\mathbb{S}$-submodule N is generated by elements of the form $(\dots, 0, \mathbf{s}, -\mathbf{s}A, 0, \dots)$, with $\mathbf{s}$ in say the ith coordinate. This element is identified with $\mathbf{s}t^i - \mathbf{s}At^{i+1} = \mathbf{s}t^i(I - tA)$. It follows that $N = \mathbb{S}[t^{\pm}]^d(I - tA)$. Hence

$$G_{\mathbb{S}}(A) = (\oplus_{\mathbb{Z}} \mathbb{S}^d)/N \cong \mathbb{S}[t^{\pm}]^d / \mathbb{S}[t^{\pm}]^d (I - tA)$$

as $\mathbb{S}[t^{\pm}]$-modules. $\square$

Note that $I = tA$ on $\operatorname{coker}(I - tA)$. Hence under the isomorphisms $\operatorname{coker}(I - tA) \cong G_{\mathbb{S}}(A) \cong \widetilde{G}_{\mathbb{S}}(A)$ from the previous two lemmas, the action of t^{-1} on $\operatorname{coker}(I - tA)$ corresponds to the action of $\widetilde{A}$ on $\widetilde{G}_{\mathbb{S}}(A)$. This explains our earlier definition of the $\mathbb{S}[t^{\pm}]$-module structure on $\widetilde{G}_{\mathbb{S}}(A)$.

We next highlight the measure of the complexity of $G_{\mathbb{S}}(A)$ which was used in the preamble to this section.

Definition 8.5. Let A be a matrix over $\mathbb{S}$. Define $g_{\mathbb{S}}(A)$ to be the size of the smallest generating set for $G_{\mathbb{S}}(A)$ as an $\mathbb{S}[t^{\pm}]$-module.

Suppose that A is $d \times d$. Since $\mathbb{S}[t^{\pm}]^d$ is generated by d elements over $\mathbb{S}[t^{\pm}]$, and since $(G_{\mathbb{S}}(A)$ is a quotient of $\mathbb{S}[t^{\pm}]^d$ by Lemma 8.4, it follows that $g_{\mathbb{S}}(A) \leqslant d$. When $\mathbb{S} = \mathbb{K}$ is a field, then $g_{\mathbb{K}}(A)$ is simply the number of blocks in the rational canonical form of $A^{\times}$ over $\mathbb{K}$. Also, if $\mathbb{K}$ is the quotient field of $\mathbb{S}$ then any set which generates $G_{\mathbb{S}}(A)$ over $\mathbb{S}[t^{\pm}]$ will generate $G_{\mathbb{K}}(A)$ over $\mathbb{K}[t^{\pm}]$, so that $g_{\mathbb{K}}(A) \leqslant g_{\mathbb{S}}(A)$. However, this inequality can be strict.

Example 8.6. Let B be a $d \times d$ cycle permutation matrix, and $A = I + 2B$. Since the eigenvalues of A are distinct, it follows that A is similar over $\mathbb{Q}$ to the companion matrix of its characteristic polynomial, so that $g_{\mathbb{Q}}(A) = 1$.

Consider the map

$$\phi : \mathbb{Z}[t^{\pm}]^d / \mathbb{Z}[t^{\pm}]^d (I - tA) \to \mathbb{Z}^d / \mathbb{Z}^d (I - A)$$

induced by $\phi(t) = 1$. Any set of $\mathbb{Z}[t^{\pm}]$ generators for $G_{\mathbb{Z}}(A)$ maps to a spanning set for the $(\mathbb{Z}/2\mathbb{Z})$-vector space

$$\mathbb{Z}^d / \mathbb{Z}^d (I - A) = \mathbb{Z}^d / \mathbb{Z}^d (-2B) \cong (\mathbb{Z}/2\mathbb{Z})^d.$$

This shows that $g_{\mathbb{Z}}(A) \geqslant d$. Our remarks above show that $g_{\mathbb{Z}}(A) \leqslant d$, so that $g_{\mathbb{Z}}(A) = d$.

We now turn to polynomial matrices. Let $P(t)$ be an $r \times r$ matrix over $\mathbb{S}[t]$, and $P(t)^{\natural}$ be the $n \times n$ matrix resulting from the $\natural$-construction. Proposition 3.3 and Lemma 8.4 suggest introducing the $\mathbb{S}[t^{\pm}]$-module $G_{\mathbb{S}}(P(t))$ defined by

$$G_{\mathbb{S}}(P(t)) = \mathbb{S}[t^{\pm}]^r / \mathbb{S}[t^{\pm}]^r (I - tP(t)) = \operatorname{coker}(I - tP(t)).$$

Lemma 8.7. *$G_{\mathbb{S}}(P(t))$ and $G_{\mathbb{S}}(P(t)^{\natural})$ are isomorphic $\mathbb{S}[t^{\pm}]$-modules.*

Proof. Recall from Section 3 that $P(t)^{\natural}$ is indexed by symbols j_k, where $1 \leqslant j \leqslant r$ and $0 \leqslant k \leqslant d(j)$. Let $\mathbf{e}_{j_k} \in \mathbb{S}[t^{\pm}]^n$ be the corresponding elementary basis vector, and similarly $\mathbf{e}_j \in \mathbb{S}[t^{\pm}]^r$. Define $\phi : \mathbb{S}[t^{\pm}]^n \to \mathbb{S}[t^{\pm}]^r$ by $\phi(\mathbf{e}_{j_k}) = t^k \mathbf{e}_j$. Then for $1 \leqslant k \leqslant d(j)$,

$$\phi\big[\mathbf{e}_{j_k}(I - t\{P(t)^{\natural}\})\big] = \phi(\mathbf{e}_{j_k}) - t\phi(\mathbf{e}_{j_{k-1}}) = 0,$$

while

$$\phi\big[\mathbf{e}_{j_0}(I - t\{P(t)^\natural\})\big] = \mathbf{e}_j - tp_{j1}(t)\mathbf{e}_1 - \cdots - tp_{jr}(t)\mathbf{e}_r = \mathbf{e}_j(I - tP(t)).$$

Hence

$$\phi\big[\mathbb{S}[t^\pm]^n(I - t\{P(t)^\natural\})\big] = \mathbb{S}[t^\pm]^r(I - tP(t)).$$

This shows that ϕ induces an isomorphism of $\mathbb{S}[t^\pm]$-modules

$$G_\mathbb{S}(P(t)^\natural) \cong \operatorname{coker}(I - t\{P(t)^\natural\}) \xrightarrow{\phi} \operatorname{coker}(I - tP(t)) \cong G_\mathbb{S}(P(t))$$

completing the proof. □

Since $\operatorname{coker}(I - tP(t))$ is generated by the images of the r elementary basis vectors, it follows that $g_\mathbb{S}(P(t)^\natural) \leqslant r$, although $P(t)^\natural$ may have size much larger than r.

Suppose that A is a matrix over $\mathbb{S}$, and that $P(t)$ is an $r \times r$ polynomial matrix such that there is a $\mathbb{S}[t^\pm]$-homomorphism from $G_\mathbb{S}(P(t)^\natural)$ onto $G_\mathbb{S}(A)$. Then $g_\mathbb{S}(A) \leqslant g_\mathbb{S}(P(t)^\natural) \leqslant r$, so that $g_\mathbb{S}(A)$ is a lower bound for the size of any such polynomial matrix. Our final result shows that, even with a further Perron restriction, we can always come within one of this lower bound.

Theorem 8.8. *Let A be a Perron matrix over $\mathbb{S}$. Then there exists a polynomial matrix $P(t)$ over $\mathbb{S}_+[t]$ of size at most $g_\mathbb{S}(A) + 1$ such that $\rho(P(t)^\natural) = \rho(A)$, $P(t)^\natural$ is primitive, and there is a $\mathbb{S}[t^\pm]$-module homomorphism from $G_\mathbb{S}(P(t)^\natural)$ onto $G_\mathbb{S}(A)$.*

Proof. Suppose that A is a $d \times d$ Perron matrix over $\mathbb{S}$. As before, let $\mathbb{K}$ denote the quotient field of $\mathbb{S}$. Let $\lambda = \rho(A) > 0$ be the spectral radius of A, and $\mathbf{v}$ be an eigenvector corresponding to λ. Let m be the dimension of the eventual range $\mathscr{R}(A)$ of A. Set $V = \mathbb{R}\mathbf{v}$, and define $\pi_V : \mathscr{R}(A) \to V$ to be projection to V along the direct sum of the generalized eigenspaces of the other eigenvalues of $A^\times = A|_{\mathscr{R}(A)}$. Identifying V with $\mathbb{R}$ via $t\mathbf{v} \leftrightarrow t$ means we can think of π_V as having range $\mathbb{R}$.

Let $g = g_\mathbb{S}(A)$. We identify $G_\mathbb{S}(A)$ with $\widetilde{G}_\mathbb{S}(A)$, and for notational simplicity use A instead of $\widetilde{A}$. By definition there are elements $\mathbf{x}_1, \ldots, \mathbf{x}_g \in G_\mathbb{S}(A)$ that generate $G_\mathbb{S}(A)$ over $\mathbb{S}[t^\pm]$. Since $\mathscr{R}(A) \cap \mathbb{S}^d \subset G_\mathbb{S}(A)$ spans $\mathscr{R}(A) \cap \mathbb{K}^d$ using $\mathbb{K}$-linear combinations, there must be at least one $\mathbf{x}_j$ with $\pi_V(\mathbf{x}_j) \neq 0$. Replacing $\mathbf{x}_j$ with $-\mathbf{x}_j$ if necessary, we can assume that $\pi_V(\mathbf{x}_j) > 0$. Then by adding to each $\mathbf{x}_i$ a large enough integral multiple of $\mathbf{x}_j$, we can also assume that $\pi_V(\mathbf{x}_i) > 0$ for $1 \leqslant i \leqslant g$.

For a finite set $\mathscr{W}$ of vectors in $\mathbb{R}^d$, let $K(\mathscr{W}) = \sum_{\mathbf{w} \in \mathscr{W}} \mathbb{R}_+\mathbf{w}$ denote the nonnegative real cone generated by $\mathscr{W}$.

Since $G_\mathbb{S}(A)$ spans $\mathscr{R}(A) \cap \mathbb{K}^d$ using $\mathbb{K}$-linear combinations, for all sufficiently large D the cone

$$K\big(\{\mathbf{x}_i A^j : 1 \leqslant i \leqslant g, -D \leqslant j \leqslant D\}\big)$$

has nonempty interior in $\mathscr{R}(\mathscr{A})$. We extract vectors $\mathbf{b}_1, \ldots, \mathbf{b}_{m-1}$ from $\{\mathbf{x}_i A^j : 1 \leqslant i \leqslant g, -D \leqslant j \leqslant D\}$ such that $\{\mathbf{b}_1, \ldots, \mathbf{b}_{m-1}, \mathbf{v}\}$ is linearly independent. Proceeding as in the construction of $\mathbf{u}$ in Theorem 5.1, we choose $c_0, c_1, \ldots, c_{m-1}$ from $\mathbb{K}_{++}$ to define

$$\mathbf{x}_{g+1} = c_0\mathbf{v} + c_1(\mathbf{v} - \mathbf{b}_1) + \ldots + c_{m-1}(\mathbf{v} - \mathbf{b}_{m-1})$$

such that $\pi_V(\mathbf{x}_{g+1}) > 0$ and $\mathbf{v}$ is in the interior of $K(\{\mathbf{b}_1, \ldots, \mathbf{b}_{m-1}, \mathbf{x}_{g+1}\})$. Define $\mathbf{b}_m = \mathbf{x}_{g+1}$. Applying a large power of A and adjusting D of necessary, we may assume that each $\mathbf{b}_j \in \mathbb{S}^d$. Set

$$\mathscr{X} = \left\{\mathbf{x}_i A^j : 1 \leqslant i \leqslant g+1, -D \leqslant j \leqslant D\right\}$$

and $\mathscr{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$.

Let $\pi_{\mathscr{R}(A)} : \mathbb{R}^d \to \mathscr{R}(A)$ denote the projection to $\mathscr{R}(A)$ along the eventual nullspace of A. For each standard basis vector $\mathbf{e}_j \in \mathbb{R}^d$ let $\mathbf{u}_j = \pi_{\mathscr{R}(A)}(\mathbf{e}_j)$. Observe that $\mathbf{u}_j = (\mathbf{e}_j A^d)(A^\times)^{-d}$, so $\mathbf{u}_j \in G_{\mathbb{S}}(A)$ for every j. Since the $\mathbf{x}_i$ generate under $\mathbb{S}[t^\pm]$, by increasing D if necessary one last time we may assume there are $\gamma_j(\mathbf{x}) \in \mathbb{S}$ such that $\mathbf{u}_j = \sum_{\mathbf{x}\in\mathscr{X}} \gamma_j(\mathbf{x})\mathbf{x}$. Set

$$\Gamma = \sum_{j=1}^{d} \sum_{\mathbf{x}\in\mathscr{X}} |\gamma_j(\mathbf{x})|.$$

We claim that for any $\mathbf{v} \in G_{\mathbb{S}}(A) \cap \mathbb{S}^d$,

$$\mathbf{v} = \sum_{\mathbf{x}\in\mathscr{X}} \gamma(\mathbf{x})\mathbf{x}, \qquad \text{where } \gamma(\mathbf{x}) \in \mathbb{S} \text{ and } |\gamma(\mathbf{x})| \leqslant \Gamma\|\mathbf{v}\|_\infty \text{ for all } \mathbf{x} \in \mathscr{X}. \tag{8.2}$$

To check this claim, suppose that $\mathbf{v} = \sum_{j=1}^d v_j \mathbf{e}_j \in G_{\mathbb{S}}(A) \cap \mathbb{S}^d$, where $v_j \in \mathbb{S}$ and $|v_j| \leqslant \|\mathbf{v}\|_\infty$ for $1 \leqslant j \leqslant d$. Then

$$\begin{aligned}
\mathbf{v} = \pi_{\mathscr{R}(A)}(\mathbf{v}) &= \sum_{j=1}^{d} v_j \pi_{\mathscr{R}(A)}(\mathbf{e}_j) = \sum_{j=1}^{d} v_j \mathbf{u}_j \\
&= \sum_{j=1}^{d} v_j \left(\sum_{\mathbf{x}\in\mathscr{X}} \gamma_j(\mathbf{x})\mathbf{x}\right) = \sum_{\mathbf{x}\in\mathscr{X}} \left(\sum_{j=1}^{d} v_j \gamma_j(\mathbf{x})\right) = \sum_{\mathbf{x}\in\mathscr{X}} \gamma(\mathbf{x})\mathbf{x},
\end{aligned}$$

where

$$|\gamma(\mathbf{x})| = \left|\sum_{j=1}^{d} v_j \gamma_j(\mathbf{x})\right| \leqslant \Gamma\|\mathbf{v}\|_\infty \quad \text{for all } \mathbf{x} \in \mathscr{X},$$

establishing (8.2).

Our goal now is to show that if $\mathbf{z} \in G_{\mathbb{S}}(A)$ with $\pi_V(\mathbf{z}) > 0$, then for all sufficiently large $N > D$ we can write $\mathbf{z}A^N$ as an $\mathbb{S}_{++}$-combination of vectors from $\mathscr{X}$. Applying this to $\mathbf{z} = \mathbf{x}_1, \ldots, \mathbf{z} = \mathbf{x}_{g+1}$ puts us into the situation of Section 4, and the construction of the required polynomial matrix $P(t)$ of size $g+1$ then follows using the same method as in the proof of Theorem 5.1. As in the proof of Theorem 5.1, we consider separately the cases $\mathbb{S}=\mathbb{Z}$ and $\mathbb{S}$ dense.

First suppose that $\mathbb{S}=\mathbb{Z}$. Then $|\det A^{\times}| = \prod_{\mu\in \mathrm{sp}^{\times}(A)} |\mu| \in \mathbb{Z}_{++}$, and hence $\lambda \geqslant 1$. If $\lambda = 1$, then since A is Perron we must have that $\mathrm{sp}^{\times}(A)=\{1\}$ and $G_{\mathbb{Z}}(A)=\mathscr{R}(A)\cap\mathbb{Z}^d\cong\mathbb{Z}$. In this case simply take $P(t)=[1]$.

Now suppose that $\lambda>1$. The lattice $\bigoplus_{j=1}^m \mathbb{Z}\mathbf{b}_j$ has a fundamental domain $F=\bigoplus_{j=1}^m [0,1)\mathbf{b}_j$. Let $C=\max\{\|\mathbf{w}\|_\infty : \mathbf{w}\in F\}$. Choose $\Delta\in\mathbb{Z}_{++}$ such that $\Delta>2C\Gamma$ and $\Delta\mathbf{x}\in\mathbb{Z}^d$ for all $\mathbf{x}\in\mathscr{X}$. Put $\mathbf{y}=\Delta\sum_{\mathbf{x}\in\mathscr{X}}\mathbf{x}\in G_{\mathbb{Z}}(A)\cap\mathbb{Z}^d$.

Suppose that $\mathbf{z}\in G_{\mathbb{Z}}(A)$ and $\pi_V(\mathbf{z})>0$. Since $\lambda>1$ and $\mathbf{v}\in K(B)^\circ$, for all sufficiently large N we have that $\mathbf{z}A^N-\mathbf{y}\in K(\mathscr{B})^\circ$. Hence there are $n_j\in\mathbb{Z}_+$ such that

$$\mathbf{z}A^N-\mathbf{y}=\sum_{j=1}^d n_j\mathbf{b}_j+\mathbf{w},$$

where $\mathbf{w}\in F$ and so $\|\mathbf{w}\|_\infty\leqslant C$. Since $\mathbf{z}A^N$, $\mathbf{y}$, and the $\mathbf{b}_j$ are in $G_{\mathbb{Z}}(A)\cap\mathbb{Z}^d$, it follows that $\mathbf{w}\in G_{\mathbb{Z}}(A)\cap\mathbb{Z}^d$. By (8.2), $\mathbf{w}=\sum_{\mathbf{x}\in\mathscr{X}}\gamma(\mathbf{x})\mathbf{x}$, where $\gamma(\mathbf{x})\in\mathbb{Z}$ and $|\gamma(\mathbf{x})|\leqslant\Gamma\|\mathbf{w}\|_\infty\leqslant C\Gamma$ for all $\mathbf{x}\in\mathscr{X}$. Thus

$$\mathbf{z}A^n=\sum_{j=1}^d n_j\mathbf{b}_j+\sum_{\mathbf{x}\in\mathscr{X}}[\Delta+\gamma(\mathbf{x})]\mathbf{x}.$$

Since $\mathscr{B}\subset\mathscr{X}$ and $\Delta>|\gamma(\mathbf{x})|$ for all $\mathbf{x}\in\mathscr{X}$, we have that

$$\mathbf{z}A^N=\sum_{\mathbf{x}\in\mathscr{X}}\xi(\mathbf{x})\mathbf{x},$$

where $\xi(\mathbf{x})\in\mathbb{Z}_{++}$ for all $\mathbf{x}\in\mathscr{X}$. This completes the case $\mathbb{S}=\mathbb{Z}$.

Finally, suppose that $\mathbb{S}$ is dense in $\mathbb{R}$. Let $\mathbf{z}\in G_{\mathbb{S}}(A)$ with $\pi_V(\mathbf{z})>\mathbf{0}$. Then for all sufficiently large N we have that $\mathbf{z}A^N\in\mathbb{S}^d$ and $\mathbf{z}A^N\in K(\mathscr{B})^\circ$. Since $\mathbb{S}$ is dense, we can find $\delta\in\mathbb{S}_{++}$ such that $\delta\mathbf{x}\in\mathbb{S}^d$ for all $\mathbf{x}\in\mathscr{X}$ and also that

$$\mathbf{z}A^N-\delta\sum_{\mathbf{x}\in\mathscr{X}}\mathbf{x}\in K(\mathscr{B})^\circ.$$

By density of $\mathbb{S}$, we can choose $s_j\in\mathbb{S}_+$ such that

$$\mathbf{z}A^N-\delta\sum_{\mathbf{x}\in\mathscr{X}}\mathbf{x}=\sum_{j=1}^m s_j\mathbf{b}_j+\mathbf{w},$$

where $\|\mathbf{w}\|_\infty<\delta/2\Gamma$. Then $\mathbf{w}\in G_{\mathbb{S}}(A)\cap\mathbb{S}^d$, and so by (8.2) we have that $\mathbf{w}=\sum_{\mathbf{x}\in\mathscr{X}}\gamma(\mathbf{x})\mathbf{x}$, where $\gamma(\mathbf{x})\in\mathbb{S}$ and $|\gamma(\mathbf{x})|\leqslant\Gamma\|\mathbf{w}\|_\infty\leqslant\delta/2$ for all $\mathbf{x}\in\mathscr{X}$. Thus

$$\mathbf{x}A^N=\sum_{j=1}^m s_j\mathbf{b}_j+\sum_{\mathbf{x}\in\mathscr{X}}[\delta+\gamma(\mathbf{x})]\mathbf{x}.$$

Since $\mathscr{B} \subset \mathscr{X}$ and $\delta > |\gamma(\mathbf{x})|$ for all $\mathbf{x} \in \mathscr{X}$, we have that

$$\mathbf{z}A^N = \sum_{\mathbf{x}\in\mathscr{X}} \xi(\mathbf{x})\mathbf{x},$$

where $\xi(\mathbf{x}) \in \mathbb{S}_{++}$ for all $\mathbf{x} \in \mathscr{X}$. $\square$

Example 8.9. It is not possible to strengthen the statement of Theorem 8.8 by simply replacing $g_{\mathbb{S}}(A) + 1$ with $g_{\mathbb{S}}(A)$. For let A be the companion matrix of $p(t) = t^2 - 3t + 1$ and $\mathbb{S} = \mathbb{Z}$. Clearly $g_{\mathbb{S}}(A) = 1$. Now suppose $P(t)^\natural$ is primitive and there is an $\mathbb{S}[t^\pm]$-module homomorphism from $G_{\mathbb{S}}(P(t)^\natural)$ onto $G_{\mathbb{S}}(A)$. Then the two positive roots of $p(t)$ must be contained in the eigenvalues of $P(t)^\natural$, and therefore the size of $P(t)$ must be greater than 1 by Section 7.

Remark 8.10. In Theorem 8.8 we considered possibly singular matrices A. This is necessary: when $\mathbb{S}$ is not a principal ideal domain, it can happen for a singular matrix A over $\mathbb{S}$ there is no nonsingular matrix B over $\mathbb{S}$ such that the $\mathbb{S}[t^\pm]$-modules $G_{\mathbb{S}}(A)$ and $G_{\mathbb{S}}(B)$ are isomorphic [7, Proposition 2.1].

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