Reprinted from JOURNAL OF NUMBER THEORY
Vol. 40, No. 2, February 1992

Matrices of Perron Numbers*

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Communicated by H. L. Montgomery

Received November 18, 1989; revised November 14, 1990

A Perron number is an algebraic integer $\geqslant 1$ that is strictly greater than the absolute value of its other conjugates. These numbers are precisely the spectral radii of nonnegative aperiodic integral matrices, and they possess an interesting arithmetic much like the natural numbers. Motivated by applications to symbolic dynamics and coding theory, we prove that the spectral radius of an aperiodic matrix whose nonzero entries are Perron is also a Perron number. Thus the set of Perron numbers is the "closure" of the natural numbers under the operation of taking spectral radii of aperiodic matrices with nonzero entries in the set. For a given nonnegative aperiodic integral matrix, we also obtain, by use of Diophantine arguments, an explicit upper bound on the smallest eigenvector in the dominant direction in which every entry is a Perron number. © 1992 Academic Press, Inc.

1. INTRODUCTION

A *Perron number* is an algebraic integer $\geqslant 1$ whose remaining conjugates have strictly smaller absolute value. Denote the set of Perron numbers by $\mathbb{P}$. Call a nonnegative matrix *aperiodic* if some positive power of it is strictly positive. We showed in [L, Thm. 1] that $\lambda \in \mathbb{P}$ if and only if λ is the spectral radius of an aperiodic nonnegative integral matrix, and this provided a simple characterization of the values for the topological entropy for mixing shifts of finite type. We also showed that λ is the spectral radius of a general nonnegative integral matrix if and only if $\lambda = 0$ or $\lambda^k \in \mathbb{P}$ for some $k \geqslant 1$ [L, Thm. 3].

The set $\mathbb{P}$ of Perron numbers is closed under addition and multiplication, and has an arithmetic much like the natural numbers $\mathbb{N}$. In particular, there are Perron numbers that are *irreducible* in the sense that they have no nontrivial factorizations in $\mathbb{P}$, and every Perron number is the product of finitely many irreducibles. Unfortunately, factorization into irreducibles

* The author gratefully acknowledges support by NSF Grant DMS-8706284.

0022-314X/92 $3.00

is not necessarily unique, but there are at most a finite number of distinct factorizations [L, Thm. 4]. Boyd [B, Sect. 3] has computed the smallest Perron number of degree d for $d \leqslant 12$, and formulated a general conjecture that states, in part, that this number is the root of $x^d - x - 1$ provided that $d \not\equiv 3$ or 5 (mod 6).

Since $\mathbb{P}$ can be regarded as a generalization of $\mathbb{N}$, it is natural to investigate the spectral radii of matrices whose entries are in $\mathbb{P}_0 = \mathbb{P} \cup \{0\}$. This question has also arisen in connection with symbolic dynamics [T], and Tuncel's work on Bernoulli quotients of Bernoulli shifts was our main motivation for the results of Section 2. In Theorem 1 we prove that no additional numbers arise in this process, i.e., that an aperiodic matrix with entries in $\mathbb{P}_0$ has spectral radius in $\mathbb{P}$. This means that $\mathbb{P}$ is "closure" of $\mathbb{N}$ with respect to the operation of taking spectral radii of aperiodic matrices whose nonzero entries are the set. This closure property provides evidence that $\mathbb{P}$ is a natural object of study in its own right. The analogous closure statement for general (not necessarily aperiodic) matrices is the content of Theorem 2.

If A is an $r \times r$ aperiodic nonnegative integral matrix whose spectral radius is an integer n, then A has an eigenvector $x \in \mathbb{N}^r$ with eigenvalue n. We may assume that the entries of x have no common factor, so that x is the smallest integral eigenvector. This eigenvector is used in symbolic dynamics [M] and in coding theory [ACH] for the basic state splitting algorithm. The complexity of the algorithm depends on the size of x. Motivated by these considerations, Ashley [A] showed that the components of x are bounded above by n^{r-1}, and that this bound is sharp. In general, the spectral radius λ of a nonnegative integral matrix A is no longer integral, but is in $\mathbb{P}$. In this case, there is an eigenvector $x \in \mathbb{P}^r$, whose entries we can assume have no common factor in $\mathbb{P}$. There may be several such vectors, since Perron factorizations may not be unique (cf. Example 1). If $M = \max_{i,j} A_{ij}$, we prove in Theorem 3 that there exists a Perron eigenvector x for A with

$$\|x\|_\infty \leqslant (rM)^{(rM)^{16r^3}}.$$

An explicit estimate along these lines is also possible if A has entries in $\mathbb{P}_0$, but the form is more complicated.

2. Spectral Radii of Matrices of Perron Numbers

We show that aperiodic matrices with entries in $\mathbb{P}_0 = \mathbb{P} \cup \{0\}$ have spectral radius in $\mathbb{P}$. We then establish the "weak" analogue of this.

THEOREM 1. *Suppose that $A=[\alpha_{ij}]$ is aperiodic, and that every $\alpha_{ij}\in\mathbb{P}_0$. Then the spectral radius λ_A of A is Perron.*

Proof. Let $K=\mathbb{Q}(\{\alpha_{ij}\colon 1\leqslant i,j\leqslant r\})$. The characteristic polynomial $\chi_A(t)$ of A is in $K[t]$. Let L be a finite Galois extension of $\mathbb{Q}$ containing K and the roots of $\chi_A(t)$. Every conjugate of λ_A has the form $\sigma(\lambda_A)$ for some $\sigma\in\mathrm{Gal}(L/\mathbb{Q})$. We show that if $\sigma(\lambda_A)\neq\lambda_A$, then $|\sigma(\lambda_A)|<\lambda_A$, so that $\lambda_A\in\mathbb{P}$.

First suppose that $\sigma\in\mathrm{Gal}(L/\mathbb{Q})$ fixes K. Then $\sigma(\chi_A(t))=\chi_A(t)$, so that $\sigma(\lambda_A)$ is a root of $\chi_A(t)$. Since A is nonnegative and aperiodic, the Perron–Frobenius theorem shows that either $\sigma(\lambda_A)=\lambda_A$, or $|\sigma(\lambda_A)|<\lambda_A$, completing the argument in this case.

If σ does not fix K, then $\sigma(\alpha_{pq})\neq\alpha_{pq}$ for some choice of p, q. Since α_{pq} is Perron, it follows that $|\sigma(\alpha_{pq})|<\alpha_{pq}$. Denote the spectral radius of a matrix B by $\rho(B)$. Since $\sigma(\lambda_A)$ is an eigenvalue of $\sigma(A)$, we have that $|\sigma(\lambda_A)|\leqslant\rho(\sigma A)$. If $|\sigma A|$ denotes the matrix $[|\sigma\alpha_{ij}|]$, then $\rho(\sigma A)\leqslant\rho(|\sigma A|)$ [G, XIII.2, Lemma 2]; and since $|\sigma A|<A$, we have that $\rho(|\sigma A|)<\rho(A)=\lambda_A$ [S, Thm. 1.1(e)]. Thus $|\sigma(\lambda_A)|<\lambda_A$ in this case, which completes the proof. ∎

We call an algebraic integer λ *weak Perron* if all of the conjugates of λ have absolute value $\leqslant\lambda$. Denote the set of weak Perron numbers by $\tilde{\mathbb{P}}$. Perron–Frobenius theory shows that $\lambda\in\tilde{\mathbb{P}}$ if and only if $\lambda^k\in\mathbb{P}$ for some $k\geqslant 1$ [L, Sect. 4]. As we remarked in Section 1, $\tilde{\mathbb{P}}_0=\tilde{\mathbb{P}}\cup\{0\}$ is exactly the set of spectral radii of all matrices over $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The following shows that $\tilde{\mathbb{P}}_0$ is the closure of $\mathbb{N}_0$ with respect to taking spectral radii.

THEOREM 2. *If $A=[\alpha_{ij}]$ is an arbitrary matrix with $\alpha_{ij}\in\tilde{\mathbb{P}}_0$, then $\lambda_A\in\tilde{\mathbb{P}}_0$.*

Proof. We use the notations from the proof of Theorem 1. If $\sigma\in\mathrm{Gal}(L/\mathbb{Q})$, then $\sigma(\lambda_A)$ is an eigenvalue of σA. Since $|\sigma(\alpha_{ij})|\leqslant\alpha_{ij}$, it follows that

$$|\sigma(\lambda_A)|\leqslant\rho(\sigma A)\leqslant\rho(|\sigma A|)\leqslant\rho(A)=\lambda_A,$$

proving that $\lambda_A\in\tilde{\mathbb{P}}_0$. ∎

3. Perron Eigenvectors

In this section we establish an upper bound on the size of a Perron eigenvector for an aperiodic nonnegative integral matrix A. By cancelling any common Perron factors from the entries, we may assume the eigenvector has no common Perron factors among its entries. However, unlike the case of integral spectral radius, this does not uniquely determine the eigenvector.

EXAMPLE 1. Let $\alpha = (1+\sqrt{5})/2$, and

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}, \qquad x = \begin{bmatrix} \alpha+2 \\ \alpha \end{bmatrix}, \qquad y = \begin{bmatrix} 5\alpha \\ \alpha+2 \end{bmatrix}.$$

Then x and y are both eigenvectors for A with eigenvalue $\lambda_A = \alpha^3$, and both entries in each eigenvector are irreducible Perron numbers [L, Sect. 5]. The basis of this example is the existence of distinct factorizations $5 \cdot \alpha \cdot \alpha = (\alpha+2)(\alpha+2)$ of a Perron number into irreducibles.

Despite this lack of uniqueness, we are able to find an upper bound on the entries of some Perron eigenvector for A. If $x \in \mathbb{R}^r$, put $\|x\|_\infty = \max_{1 \leqslant j \leqslant r} |x_j|$.

THEOREM 3. *Suppose that A is an $r \times r$ aperiodic nonnegative integral matrix, with Perron eigenvalue λ and maximal element $M = \max_{i,j} A_{ij}$. Then there exists an eigenvector x for T with eigenvalue λ, whose components are in $\mathbb{P}$, and such that*

$$\|x\|_\infty \leqslant (rM)^{(rM)^{16r^3}}.$$

The proof exploits the fact that the columns of the adjoint matrix of $\lambda I - A$ are strictly positive eigenvectors for A with eigenvalue λ. Although entries in the adjoint matrix are positive and are polynomials in λ, they need not be Perron since the polynomials will in general have some negative coefficients. We then use ideas from Diophantine analysis, similar to those found in [St, Chap. 4], to obtain an upper bound on the power K of λ needed so that each entry of the adjoint matrix becomes Perron when multiplied by λ^K.

In what follows, the *height* of a polynomial is the maximum of the absolute values of its coefficients.

LEMMA 1. *Let $\alpha \in \mathbb{P}$ have degree d, and suppose that $f(t) \in \mathbb{Z}[t]$ has degree D and height H. If $f(\alpha) \neq 0$, then*

$$|f(\alpha)| \geqslant [\alpha^D (D+1) H]^{-d+1}.$$

Proof. Denote the conjugates of α by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. Then $f(\alpha_j) \neq 0$ for every j, so that $\prod_{j=1}^{d} f(\alpha_j)$ is a nonzero integer. Since $|\alpha_j| \leqslant \alpha$ for every j, clearly

$$|f(\alpha_j)| \leqslant \alpha^D (D+1) H. \tag{3-1}$$

Thus

$$1 \leqslant \left| \prod_{j=1}^{d} f(\alpha_j) \right| = |f(\alpha)| \, [\alpha^D(D+1)\,H]^{d-1}. \quad \blacksquare$$

LEMMA 2. *Let $\alpha \in \mathbb{P}$ have degree $d \geqslant 2$ and conjugates $\alpha_1 = \alpha, \alpha_2, ..., \alpha_d$. Then for $k \geqslant 2$ we have that*

$$\frac{|\alpha_k|}{\alpha} < 1 - (\alpha d)^{-6d^3}.$$

Proof. Put $g(t) = \prod_{i,j=1}^{d} (t - \alpha_i \alpha_j) \in \mathbb{Z}[t]$. Then

$$|g'(\alpha^2)| = \prod_{(i,j) \neq (1,1)} |\alpha^2 - \alpha_i \alpha_j| \neq 0.$$

If $2 \leqslant k \leqslant d$, there is a pair $(p, q) \neq (1, 1)$ with $\alpha_p \alpha_q = |\alpha_k|^2$. Since $|\alpha^2 - \alpha_i \alpha_j| \leqslant 2\alpha^2$ for all i and j, it follows that

$$|g'(\alpha^2)| \leqslant (\alpha^2 - |\alpha_k|^2)(2\alpha^2)^{d^2}. \tag{3-2}$$

We now apply Lemma 1 to $f(t) = g'(t)$ and α^2. Here $\deg(\alpha^2) \leqslant d$, $D = \deg(g') = d^2 - 1$, and since $g(t)$ is majorized by $(t + \alpha^2)^{d^2}$ we have that

$$H = H(g') \leqslant \deg(g)\, H(g) \leqslant d^2 2^{d^2} (\alpha^2)^{d^2}.$$

Using that $d \geqslant 2$, Lemma 1 implies that

$$|g'(\alpha^2)| \geqslant \alpha^{-4d^3} d^{-5d^3}.$$

This, combined with (3-2), shows that

$$2\alpha^{-4d^3 - 2d^2 - 2} d^{-5d^3 - d^2 - 1} \leqslant 1 - \frac{|\alpha_k|^2}{\alpha^2},$$

from which, after recalling that $d \geqslant 2$, we obtain that

$$\frac{|\alpha_k|^2}{\alpha^2} \leqslant 1 - 2(\alpha d)^{-6d^3}.$$

Since $\sqrt{1 - x} \leqslant 1 - x/2$ for small x, we obtain the desired inequality. $\blacksquare$

LEMMA 3. *Suppose that $\alpha \in \mathbb{P}$ has degree d. If $f(t) \in \mathbb{Z}[t]$ has degree D and height H, and if $f(\alpha) > 0$, then $\alpha^K f(\alpha) \in \mathbb{P}$ provided that*

$$K \geqslant [\alpha^D(D+1)\,H]^d \, (\alpha d)^{6d^3}.$$

Proof. Denote the conjugates of α by $\alpha_1 = \alpha, \alpha_2, ..., \alpha_d$. The result is trivial if $d = 1$, so we assume that $d \geqslant 2$. The conjugates of $\alpha^K f(\alpha)$ are $\alpha_j^K f(\alpha_j)$, so we require a K_0 so that if $K \geqslant K_0$, then $\alpha^K f(\alpha) > |\alpha_j^K f(\alpha_j)|$ for $2 \leqslant j \leqslant d$. This inequality is equivalent to

$$\left(\frac{|\alpha_j|}{\alpha}\right)^K < \frac{f(\alpha)}{|f(\alpha_j)|} \qquad (2 \leqslant j \leqslant d). \tag{3-4}$$

By Lemma 1 we have that

$$[\alpha^D(D+1)H]^{-d+1} \leqslant f(\alpha),$$

so that, by use of (3-1),

$$[\alpha^D(D+1)H]^{-d} \leqslant \frac{f(\alpha)}{|f(\alpha_j)|}.$$

Since $(1-\theta)^K \leqslant 1/(K\theta)$ for $0 \leqslant \theta \leqslant 1$, by Lemma 2 we obtain that

$$\left(\frac{|\alpha_j|}{\alpha}\right)^K \leqslant [1-(d\alpha)^{-6d^3}]^K < \frac{(d\alpha)^{6d^3}}{K}.$$

Thus $\alpha^K f(\alpha) \in \mathbb{P}$ provided that K satisfies (3-3). ∎

Proof of Theorem 3. Since λ satisfies the characteristic polynomial $\chi_A(t)$ of A, we have that $d = \deg(\lambda) \leqslant r$.

Let $G(t) = [g_{ij}(t)] = \operatorname{adj}(tI - A)$, where $\operatorname{adj}(B)$ denotes the classical adjoint matrix of B. By standard Perron–Frobenius theory [G, XIII.2.2 (13)], we have that $g_{ij}(\lambda) > 0$ for all i and j. Let $g_i(t)$ denote $g_{i,1}(t)$, and define $y \in \mathbb{R}^r$ by $y_i = g_i(\lambda) > 0$. Since $(\lambda I - A)\, G(\lambda) = \chi_A(\lambda) I = 0$, it follows that y is an eigenvector for A with eigenvalue λ.

Next we determine K so that $\lambda^K y \in \mathbb{P}^r$. Note that $\deg g_i \leqslant r-1$, and clearly $H(g_i) \leqslant r!\, M^r$. It follows from Lemma 3 that $\lambda^K g_i(\lambda) \in \mathbb{P}$ for $1 \leqslant i \leqslant r$ provided that

$$K \geqslant K_0 = [\lambda^r r!\, M^r r]^r\, (\lambda r)^{6r^3}.$$

Now $\lambda \leqslant rM$ [S, Cor. 1.1.1], and by combining this with $r! \leqslant r^r$ and elementary manipulations, we may replace K_0 with

$$K_1 = (rM)^{15r^3}.$$

Finally, observe that

$$|g_i(\lambda)| \leqslant r\lambda^r H(g_i) \leqslant r(rM)^r\, r!\, M^r \leqslant (rM)^{3r}.$$

Thus, putting $x = \lambda^{K_1} y$, we have that $x \in \mathbb{P}^r$, and

$$\|x\|_\infty \leqslant (rM)^{3r} \lambda^{K_1} \leqslant (rM)^{3r + (rM)^{15r^3}} \leqslant (rM)^{(rM)^{16r^3}}. \quad \blacksquare$$

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium