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Locally Compact Measure Preserving Flows

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1. INTRODUCTION

Ergodic theory originally arose from study of the time evolution of a mechanical system. If T_t denotes the transformation of the phase space of a system which takes a state of the system to its state t time units later, then on physical grounds $\{T_t: t \in R\}$ obeys the group property that $T_{s+t} = T_s T_t$, i.e., forms a flow with parameter group R. Liouville discovered that a certain natural measure on phase space is preserved by each T_t, reducing statistical mechanics to the study of asymptotic properties of flows of measure preserving transformations. This approach was simplified by discretizing time and considering only $\{T_{nt_0}: n \in Z\}$ for some fixed t_0. This has the advantage that only the iterates of a fixed transformation are in question. Starting with the ergodic theorems of Birkhoff and von Neumann in 1931, an interesting theory of measure preserving transformations has been erected. Occasionally the continuous time case was investigated, notably by Weiner [17], Ambrose [1], and Ambrose and Kakutani [2]. However, developments centered on measure preserving transformations.

In the last decade some progress has been made in classifying measure preserving transformations. Kolmogorov introduced a new numerical invariant in 1958 called entropy which was suggested by Shannon's work on information theory. This invariant distinguished between some transformations which no one could previously tell apart, and some intriguing conjectures arose. In the past years a series of papers by Ornstein has completely solved a number of major problems, including the isomorphism problem for Bernoulli shifts. A basic technique used by Ornstein in these papers is a theorem of Rokhlin on the periodic approximation of transformations.

Some recent work on statistical mechanics by Ruelle [16] and others has led to considering flows of measure preserving transformations with more general parameter groups. For example, translation in any of the three coordinate directions of an infinite continuous three dimensional Gibbs state induces a flow with parameter group R^3. It is therefore interesting, and perhaps physically important, to see how much of the theory for one transformation (alias flow on Z) extends to more general flows. Conze [3] and Katznelson and Weiss [6] have done some work in this direction.

The principal result here is the analogue of the fundamental Rokhlin theorem for aperiodic flows with parameter group R^n. This is possibly the only nontrivial step in extending Ornstein's isomorphism theorems to multidimensional Bernoulli flows. Using a metric constructed to prove Rokhlin's theorem, we next measurably partition a space acted on by an arbitrary flow on R^n into invariant components each of which is acted on aperiodically by a quotient of the flow. This is the aperiodic version of the ergodic decomposition for transformations due to Halmos [4]. Finally, we investigate factors of flows and as an application of Rokhlin's theorem prove that every flow has a factor of finite entropy.

2. Rokhlin's Theorem for n-Dimensional Flows

The stage on which our transformations will act is a nonatomic Lebesgue measure space (X, Σ, μ). This is a set X of points, a σ-algebra Σ of subsets of X, and a countably additive complete positive measure μ on Σ with $\mu(X) = 1$, and with the further property that the measure space is measure theoretically isomorphic to a measurable subset of the unit interval with Lebesgue measure. An equivalent axiomatic description of Lebesgue spaces has been given by Rokhlin [15]. Most measure spaces encountered are Lebesgue spaces, and restricting our attention to them eliminates unpleasant and needless pathology.

An invertible measure preserving transformation of (X, Σ, μ) is a bijection $S: X \to X$ such that $S(\Sigma) = \Sigma$ and $\mu(SE) = \mu(E)$ for all measurable sets E. Such transformations under composition form a group which we denote by imp(X). Rokhlin's theorem may be stated as follows. Suppose $S \in \text{imp}(X)$ is aperiodic in the sense that $\mu\{x\colon S^k x = x\} = 0$ for all k. Then given a positive integer k and positive number ϵ, there is a measurable set $F \subset X$ such that F, $SF, \ldots, S^{k-1}F$ are pairwise disjoint and $\mu(\bigcup_{i=0}^{k-1} S^i F) > 1 - \epsilon$. A proof of this fact is given in Halmos [5]. Now S can be regarded as a flow on the group of integers Z, that is, a homomorphism $i \mapsto S^i$ from Z to imp(X). Is there an analogous result for flows with parameter groups other than Z?

As a step in extending the isomorphism theorem for Bernoulli transformations, two special groups have been treated. Ornstein [11] observed that the Ambrose–Kakutani theorem [2] representing flows on the group of reals R as flows built under a function quickly yields a version for R. Conze [3] and Katznelson and Weiss [6] have given one for Z^n. Using somewhat different ideas, we give here a proof for R^n which also applies to the previous cases, and discuss its extension to locally compact parameter groups. This work is the first, and possibly only nontrivial step necessary to further extend the isomorphism theorem.

An *n-dimensional flow* $T = \{T_t: t \in R^n\}$ on X is a homomorphism $T: R^n \to \text{imp}(X)$ which has the measurability property that if E is a measurable subset of X, then $\{(x, t): T_t x \in E\}$ is measurable in the product space $X \times R^n$. T is called *aperiodic* if there is a null set $N \subset X$ such that if $x \notin N$ and $t \neq 0$, then $T_t x \neq x$. Q will denote a semi-open cube (or possibly a rectangle) in R^n centered at the origin. If $F \subset X$, then $T_Q F$ denotes $\bigcup_{t \in Q} T_t F$. Call $T_Q F$ *disjoint* if $\{T_t F\}_{t \in Q}$ is disjoint. The set F is a *Q-set* if $T_Q F$ is both disjoint and measurable in X. (Warning: There are measurable sets F for which $T_Q F$ is not measurable; e.g., let $X = [0, 1] \times [0, 1]$, $T_t(x, y) = (x, y + t) \pmod 1$ for $t \in R$, $Q = [-\frac{1}{4}, \frac{1}{4})$, and $F \subset [0, 1] \times \{0\}$ a linearly nonmeasurable set). Our version of Rokhlin's theorem for n-dimensional flows takes the following form.

THEOREM 1. *Let T be an aperiodic n-dimensional flow on X. Then for any rectangle $Q \subset R^n$ and $\epsilon > 0$, there is a set $F \subset X$ such that $T_Q F$ is disjoint, measurable, and $\mu(T_Q F) > 1 - \epsilon$. Furthermore, on $T_Q F$ the measure μ is the completed product of a measure on F with Lebesgue measure on Q.*

The last statement of the theorem means the following. There is a natural bijection $\varphi: F \times Q \to T_Q F$ defined by $\varphi(x, t) = T_t x$. We assert that there is a measure μ_F on F such that if m_Q denotes Lebesgue measure on Q, then both φ and φ^{-1} are measurable measure preserving maps between $(F \times Q, \mu_F \times m_Q)$ and $(T_Q F, \mu)$. Thus F is "really" transversal to the flow, and our intuition will not be misled.

Before starting the proof, let us deduce some convenient consequences of this result.

COROLLARY 1. *For every $x \in X$, the orbit $O(x) = \{T_t x: t \in R^n\}$ of x has measure 0.*

Proof. Fix a cube Q in R^n. Then for any $\epsilon > 0$ there is a Q-set F with $\mu(T_Q F) > 1 - \epsilon$. Since R^n contains at most a countable number of translates of Q, $F \cap O(x)$ is at most countable, so that $\mu(T_Q F \cap O(x)) = 0$. Hence for every $\epsilon > 0$ the set $O(x)$ is contained in the set

$$[T_Q F \cap O(x)] \cup (X \backslash T_Q F),$$

which has measure less than ϵ. This implies $\mu(O(x)) = 0$. Q.E.D.

If $\mathcal{N}$ denotes the σ-ideal of null sets of Σ, then the measure algebra $\Sigma/\mathcal{N}$ is a complete metric space under the distance $\rho(E, F) = \mu(E \backslash F) + \mu(F \backslash E)$. The next corollary shows that a measurable n-dimensional flow induces a jointly continuous flow on the measure algebra.

COROLLARY 2. *The map $R^n \times \Sigma/\mathcal{N} \to \Sigma/\mathcal{N}$ given by $(t, E) \mapsto T_t E$ is jointly continuous.*

Proof. Since $T_Q F$ is isomorphic to $F \times Q$, joint continuity on $T_Q F$ follows from that of translation by elements of Q of sets in $F \times Q$. The Corollary follows by observing that since $\mu(X \backslash T_Q F)$ can be made arbitrarily small, the error introduced by ignoring $X \backslash T_Q F$ can also be made arbitrarily small. Q.E.D.

The techniques here can be applied to flows with other locally compact abelian parameter groups. To be specific, let G be a separable locally compact abelian group. A measurable G-flow is a homomorphism $T: G \to \text{imp}(X)$ such that if E is a measurable subset of X, then $\{(x, g): T(g)x \in E\}$ is a measurable subset of $X \times G$. Separability of G is included to assure the measurability of some constructions. Let H be a closed subgroup of G such that G/H is compact, and Q be a Borel measurable subset of G with compact closure such that the quotient map $G \to G/H$ is bijective on Q. This means that the translates of Q by elements of H "tile" the group G. A good example to have in mind for Q is a semi-open rectangle in R^n, which tiles R^n under a discrete subgroup. In this setting, Rokhlin's theorem takes the form that if T is an aperiodic G-flow, then for any $\epsilon > 0$ there is a set $F \subset X$ such that $T_Q F$ is measurable, disjoint, and $\mu(T_Q F) > 1 - \epsilon$. Furthermore, if G is compact, we can ignore the ϵ and actually find an F so that $T_Q F$ is an invariant set with $\mu(T_Q F) = 1$. Easy modifications in the proof here shows this general statement is true for $G = R^n \times Z^m \times K$, where $n, m \geqslant 0$ and K is a finite group, and quotients of these groups.

To begin the proof, first observe that we need only prove the theorem for cubes. For suppose Q is a rectangle. We lose nothing by assuming Q is centered at the origin, say $Q = \prod_{i=1}^{n} [-r_i, r_i)$. We will almost fill a cube with disjoint translates of Q and apply the theorem to this cube. There are odd integers k_i and a real number r such that $r \geqslant k_i r_i$ and $k_1 r_1 \cdots k_n r_n > (1 - \epsilon/4) r^n$. If $Q_r = [-r, r)^n$, we can find a Q_r-set F_0 such that $\mu(T_{Q_r} F_0) > 1 - \epsilon/4$.

If

$$\Lambda = \{(2j_1 r_1, \ldots, 2j_n r_n) : j_i \in Z, |j_i| \leqslant \tfrac{1}{2}(k_i - 1), 1 \leqslant i \leqslant n\},$$

and $F = T_\Lambda F_0$, then F is a Q-set and

$$\mu(T_Q F) = \frac{m(T_\Lambda Q)}{m(Q_r)} \mu(T_{Q_r} F_0)$$
$$> \left(1 - \frac{\epsilon}{4}\right) 2^n k_1 r_1 \cdots k_n r_n / (2r)^n > \left(1 - \frac{\epsilon}{4}\right)^2 > 1 - \epsilon.$$

We now prove that "close to" any set of positive measure lies a Q-set which flows through a fixed proportion of the set.

LEMMA. *If Q is any cube in R^n and $E \subset X$ has positive measure, then there is a Q-set F such that $F \subset T_Q E$ and $\mu(E \backslash T_{3Q} F) = 0$.*

Proof of the Lemma. The main idea is to introduce a separable metric d on an invariant subset of X of full measure which is compatible with the measure structure of X. Under this metric T will be continuous, and we will exploit continuity and compactness.

We will now define d. Let $\{M_j\}_1^\infty$ be a separating sequence of measurable sets in X. This means that if x and y are distinct points of X, there is an M_j which contains exactly one of them. Let χ_j denote the characteristic function of M_j. For $j, k \geqslant 1$ and $t \in R^n$ define

$$f_{jk}(x, t) = m(B_k)^{-1} \int_{B_k} \chi_j(T_{s+t} x)\, ds,$$

where B_k is the ball of radius $1/k$ in R^n. Since T is measurable, the function $\chi_j(T_t x)$ is measurable on $X \times R^n$. Hence by Fubini's theorem, there are null sets N_j such that if $x \notin N_j$, then $\chi_j(T_t x)$ is a bounded measurable function on R^n. Observe that each N_j is invariant under T, i.e., $T_t N_j \subset N_j$ for all $t \in R^n$. Hence $X_0 = X \backslash \bigcup_1^\infty N_j$ is an invariant set of full measure. Thus, for every $x \in X_0$ and every t, $f_{jk}(x, t)$ is defined, continuous in t, and $0 \leqslant f_{jk}(x, t) \leqslant 1$. Let $B = B_1$. For $x, y \in X_0$ define

$$d(x, y) = \sum_{j,k=1}^{\infty} 2^{-j-k} \sup_{t \in B} |f_{jk}(x, t) - f_{jk}(y, t)|.$$

This is clearly a pseudometric on X_0. By Weiner's theorem [17, Theorem III''] there are null sets L_j such that

$$\lim_{k \to \infty} m(B_k)^{-1} \int_{B_k} \chi_j(T_t x)\, dt \to \chi_j(x)$$

for all $x \notin L_j$. If $L = \bigcup_1^\infty L_j$, then $\{(x, t)\colon T_t x \in L\}$ is a null set in $X_0 \times R^n$. Hence, by Fubini's theorem there is a null set $N \subset X_0$ such that if $x \notin N$, then $T_t \notin L$ for a.e. t. Clearly N is invariant, and thus $X_1 = X_0 \backslash N$ is an invariant set of full measure. We claim d is a metric on X_1. For suppose x and y are points of X_1 with $d(x, y) = 0$. There is a t_0 with $|t_0| < 1$ and $T_{t_0} x \notin L$, $T_{t_0} y \notin L$. Using Weiner's theorem together with $d(x, y) = 0$, we see $\chi_j(T_{t_0} x) = \chi_j(T_{t_0} y)$ for all j. Since the χ_j separate points, this forces $T_{t_0} x = T_{t_0} y$ and hence $x = y$. This shows d is metric. Since X_1 is an invariant set of full measure, we can assume from now on that it is the whole space X.

In order to prove the separability of (X, d), we introduce the space Y of doubly indexed sequences $\{f_{jk}(t)\}$ of continuous functions from B to $[0, 1]$. Then Y is a complete, separable metric space under the metric

$$d_Y(\{f_{jk}(t)\}, \{g_{jk}(t)\}) = \sum_{j,k=1}^{\infty} 2^{-j-k} \sup_{t \in B} |f_{jk}(t) - g_{jk}(t)|.$$

The continuity of $f_{jk}(x, t)$ in t implies that (X, d) is naturally a subspace of (Y, d_Y) under the identification $x \leftrightarrow \{f_{jk}(x, t)\}$. This establishes the separability of (X, d).

For fixed $x \in X$, the continuity of $f_{jk}(x, t)$ in t shows that the map $R^n \to (X, d)$ given by $t \mapsto T_t x$ is continuous. Actually, the map $R^n \times (X, d) \to (X, d)$ given by $(t, x) \mapsto T_t x$ is jointly continuous, but do not need this stronger statement.

As a final remark about d, we note that the balls of d are measurable. For if B_0 is the set of rational points in B, then B can be replaced by B_0 in the definition of d. This, together with the fact that $f_{jk}(x, t)$ is measurable in x for each t, shows that $d(x, y)$ is a measurable function of x.

Now let Q be any cube in R^n centered at 0, and let $E \subset X$ with $\mu(E) > 0$. We start by obtaining a Q-set F such that $F \subset T_Q E$ and

$\mu(E \cap T_Q F) > 0$. Let H be the closure of $4Q \backslash \frac{1}{2}Q$ in R^n. The idea is to find a set $D \subset E$ of positive measure which in disjoint from $T_H D$. Then on each orbit D appears in lumps spaced a certain distance apart. F will be the union of the barycenters of these lumps. The spacing will assure us that $T_Q F$ is disjoint. The way we define F will give the measurability of $T_Q F$, and the barycenter construction implies $T_Q F$ contains D, and so $T_Q F$ intersects E in a set of positive measure.

By compactness of H and continuity of the flow, $T_H x$ is a compact subset of X. Aperiodicity of T shows $x \notin T_H x$, so

$$d(x, T_H x) = \inf\{d(x, y): y \in T_H x\} > 0$$

for all $x \in E$. Since $d(x, T_H x)$ is measurable, there is a $\delta > 0$ and $E_0 \subset E$ such that $\mu(E_0) > 0$ and $d(x, T_H x) \geqslant \delta$ for $x \in E_0$. Since (X, d) is separable, it can be covered by a countable number of $\delta/4$ balls, one of which must intersect E_0 in a set of positive measure. Thus there is a $D \subset E_0$ such that $\mu(D) > 0$ and diam $(D) < \delta/2$. This shows $D \cap T_H D = \varnothing$. For suppose $x \in D \cap T_H D$, say $x = T_h y$ where $h \in H$, $y \in D$. Then $d(x, y) < \delta/2$ since $\operatorname{diam}(D) < \delta/2$, while $d(x, y) = d(T_h y, y) > \delta$ since $y \in D$.

By Weiner's theorem we can remove a null set from D in order to assume

$$\lim_{k \to \infty} m(B_k)^{-1} \int_{B_k} \chi_D(T_t x)\, dt = 1$$

for all $x \in D$. Therefore, if $x \in D$ we have

$$m\{t \in Q: T_t x \in D\} > 0 \qquad \text{and} \qquad \{t \in 3Q \backslash Q: T_t x \in D\} = \varnothing. \tag{*}$$

The second part follows from $D \cap T_H D = \varnothing$. For $x \in X$ define $D_x = \{t \in R^n: T_t x \in D\}$. Let $Q_j^- = \{t \in Q: t_j \leqslant 0\}$, $Q_j^+ = \{t \in Q: t_j \geqslant 0\}$. We now define

$$F = \{x \in X: m(D_x \cap Q_j^-) = m(D_x \cap Q_j^+),\ 1 \leqslant j \leqslant n\}.$$

This means that $x \in F$ if and only if 0 is the barycenter of $D_x \cap Q$. We will verify that F is a Q-set, $F \subset T_Q E$, and $\mu(E \cap T_Q F) > 0$.

We first show that $T_Q F$ is measurable. Let $\psi_j^{\pm}(x) = m(D_x \cap Q_j^{\pm})$, which is a measurable function of x. Note that $\psi_j^{\pm}(T_t x)$ is continuous in t. Then if Q_0 is the set of rational points in Q, we have

$$T_Q F = \bigcup_{t \in Q} T_t\{x: \psi_j^+(x) = \psi_j^-(x),\ 1 \leqslant j \leqslant n\}$$

$$= \bigcap_{m=1}^{\infty} \bigcup_{t \in Q_0} \left\{x: |\psi_j^+(T_{-t}x) - \psi_j^-(T_{-t}x)| < \frac{1}{m},\ 1 \leqslant j \leqslant n\right\},$$

which is measurable. To see the last equality, observe that by compactness of Q a continuous function on Q (in this case $\psi_j^+(T_{-t}x) - \psi_j^-(T_{-t}x)$) vanishes somewhere on Q if and only if it is arbitrarily small on Q_0. (This is essentially the proof that a function which is measurable in one variable and continuous in the other is jointly measurable). By condition (*), if $x \in F$ then $T_{3Q}x \cap F = \{x\}$. This shows that $T_Q F$ is disjoint. For if $s, t \in Q$, and $x, y \in F$ with $T_s x = T_t y$, then $y = T_{s-t}x \in T_{2Q}x$, so $y = x$, and then aperiodicity shows $s = t$. If $x \in D$, then (*) shows that $D_x \cap Q$ has a barycenter at some $t_0 \in Q$ and that $T_{t_0}x \in F$. Hence $D \subset T_Q F$, so that $\mu(E \cap T_Q F) \geqslant \mu(D) > 0$. If $x \in F$, then 0 is the barycenter of $D_x \cap Q$, a set of positive measure. Hence there is a $t_0 \in -Q$ with $T_{t_0}x \in D$, so that $x \in T_Q D$. Thus $F \subset T_Q D \subset T_Q E$.

We next use a maximality argument to show that we can reach almost every point of E by flowing F by $3Q$. Choose a Q-set F contained in $T_Q E$ which maximizes $\mu(E \cap T_Q F)$. This means that if F_0 contains F and is a Q-set contained in $T_Q E$, then $\mu(E \cap T_Q F_0) = \mu(E \cap T_Q F)$. This is possible since an increasing countable union of Q-sets is a Q-set. Let $A = E \backslash T_{3Q}F$. Suppose $\mu(A) > 0$. By replacing E by A in the argument above, we can find a Q-set $F_0 \subset T_Q A \subset T_Q E$ with $\mu(A \cap T_Q F_0) > 0$. Now $T_Q F_0 \subset T_{2Q}A$ and $T_{2Q}A \cap T_Q F = \varnothing$. Hence $F_1 = F \cup F_0$ is a Q-set contained in $T_Q E$ and containing F with

$$\mu(E \cap T_Q F_1) \geqslant \mu(E \cap T_Q F) + \mu(A \cap T_Q F_0) > \mu(E \cap T_Q F).$$

This contradiction to maximality of F proves $\mu(A) = \mu(E \backslash T_{3Q}F) = 0$, completing the proof of the lemma.

Proof of the Theorem. If we put $E = X$ in the lemma, we find that for any cube Q there is a Q-set F with $\mu(T_{3Q}F) = 1$, so that $\mu(T_Q F) \geqslant 3^{-n}$. Let

$$\alpha = \sup\{\beta: \text{for any cube } Q \text{ there is a } Q\text{-set } F \text{ with } \mu(T_Q F) \geqslant \beta\}.$$

Then $\alpha \geqslant 3^{-n}$. We will show $\alpha = 1$ by using the lemma to pick up a fixed proportion of the measure of the complement of $T_Q F$. That is, it is enough to show that if β is in the set defining α, then so is $\beta + 4^{-n}(1 - \beta)$.

Let $Q = [-r, r)^n$ be given. Let $\gamma > 0$ be a number which will be specified later. Choose k odd and so large that $m([k + 6]Q)/m(Q) < 1 + \gamma$. Applying the assumption on β to kQ, there is a kQ-set F with $\mu(T_{kQ}F) \geqslant \beta$. If $\Lambda = \{(2j_1, \ldots, 2j_n): |j_i| \leqslant \frac{1}{2}(k_i - 1)\}$, then $F_0 = T_\Lambda F$ is a Q-set and $\mu(T_Q F_0) = \mu(T_{kQ}F) \geqslant \beta$. Now

$$\mu(T_{(k+6)Q}) \leqslant \frac{m([k + 6]Q)}{m(kQ)} \mu(T_{kQ}F) < (1 + \gamma)\, \mu(T_{kQ}F).$$

Applying the lemma to $E = X \backslash T_{(k+6)Q}F$ and Q, we see there is a Q-set F_1 with $F_1 \subset T_Q E$ and $E \subset T_{3Q}F_1$ a.e. Now $T_Q F_1 \subset T_{2Q}E$ and

$$T_{2Q}E \cap T_Q F_0 = \varnothing,$$

so that $F_2 = F_0 \cup F_1$ is a Q-set. Since $E \subset T_{3Q}F_1$, we have

$$\mu(T_Q F_1) \geqslant 3^{-n}\mu(E) \geqslant 3^{-n}[1 - (1 + \gamma)\beta] > 4^{-n}(1 - \beta)$$

if γ is chosen small enough. Hence F_2 is a Q-set with

$$\mu(T_Q F_2) > \beta + 4^{-n}(1 - \beta).$$

Thus we have shown that if $\epsilon > 0$ is given, there is a Q-set F with $\mu(T_Q F) > 1 - \epsilon$.

All that remains is to prove the last statement of the theorem. We will put a measure space structure on F and check some admittedly tedious measure theoretic details.

Define $\Sigma_F = \{F_0 \subset F: T_Q F_0 \in \Sigma\}$, and for $F_0 \in \Sigma_F$ put $\mu_F(F_0) = m(Q)^{-1}\mu(T_Q F_0)$. If μ_r denotes the restriction of μ to $T_Q F$, we claim $\mu_r = \mu_F \times m_Q$, where $\times$ denotes the completed product measure. The proof that $T_Q F$ is measurable applies equally to any subrectangle $Q_0 \subset Q$ to show $T_{Q_0}F$ is μ_r measurable. If $\mathscr{M}_Q$ denotes the Lebesgue measurable subsets of Q and Σ_r the restriction of Σ to $T_Q F$, then rectangles of the form $A \times B$, $A \in \Sigma_F$ and $B \in \mathscr{M}_Q$, are μ_r measurable with

$$\mu_F \times m_Q(A \times B) = \mu_r(A \times B).$$

Hence the σ-algebra $\Sigma_F \times \mathscr{M}_Q$ generated by these rectangles is contained in Σ_r, therefore so is its completion $\Sigma_F \otimes \mathscr{M}_Q$, and on this $\mu_F \times m_Q$ and μ_r agree.

We want to show conversely that $\Sigma_r \subset \Sigma_F \otimes \mathscr{M}_Q$. Let us identify $F \times Q$ with $T_Q F$ by $\varphi(x, t) = T_t x$. The preceding paragraph shows that φ^{-1} is measurable and measure preserving. Let $Y = T_Q F \times R^n$ with measure $\mu_Y = \mu_r \times m$ on $\Sigma_r \otimes \mathscr{M}$. The transformation $S: Y \to Y$ defined by $S(x, t, u) = (x, t, t + u)$ is an invertible measure preserving transformation on Y. This follows by verification on rectangles, passing to the generated σ-algebra, and then to its completion. Let $E \subset T_Q F$ be μ_r measurable. Since T is measurable, $E^* = S\{(x, t, u): T_{t+u} x \in E\}$ is μ_Y measurable. The point about E^* is that it is independent of u, and that its t cross section $E_t{}^* = \{(x, u): (x, t, u) \in E^*\}$ equals E for every $t \in Q$. Also $E^{*u} = \{(x, t): (x, t, u) \in E^*\}$ equals $E_u \times Q = T_Q E_u$, where $E_u = T_{-u} E \cap F$. By Fubini's theorem, E^{*u} is μ_r measurable for a.e. u, so that almost every cross section E_u of E is μ_F measurable. Furthermore, we have the formula

$$\mu_r(E) = \int_Q \mu_F(E_u)\, dm_Q(u). \tag{1}$$

This follows again by Fubini's theorem since

$$m(Q)\, \mu_r(E) = \int_Q \mu_r(E_t{}^*)\, dm_Q(t) = \mu_Y(E^*) = \int_Q \mu_r(E^{*u})\, dm_Q(u)$$
$$= m(Q) \int_Q \mu_F(E_u)\, dm_Q(u).$$

We could now easily show that any set in Σ_r is equivalent to one in $\Sigma_F \otimes \mathscr{M}_Q$, but this is not quite enough to prove that φ is measurable.

Let $\Sigma_c = \{E \subset Y: E_t \text{ is } \mu_F \times m \text{ measurable for a.e. } t \in Q\}$. From the above, rectangles of $\Sigma_r \times \mathscr{M}$ are in Σ_c, and since Σ_c is a σ-algebra we have $\Sigma_r \times \mathscr{M} \subset \Sigma_c$. We claim $\Sigma_r \otimes \mathscr{M} \subset \Sigma_c$. For this it suffices to show that if $E \in \Sigma_r \times \mathscr{M}$, $\mu_Y(E) = 0$, and $D \subset E$, then $D \in \Sigma_c$. To prove this we need the fact that if $E \in \Sigma_r \times \mathscr{M}$ and $\mu_Y(E) = 0$, then $\mu_F \times m(E_t) = 0$ for a.e. $t \in Q$. For since $E \in \Sigma_c$, E_t is $\mu_F \times m$ measurable except for t in a null set N_1, and by Fubini's theorem $E_{(x,t)} = \{u: (x, t, u) \in E\}$ is m measurable except for (x, t) in a null set N_2 of μ_r measure 0. Since $\mu_r(N_2) = 0$, (1) shows that there is a null set $N_3 \subset Q$ such that if $t \notin N_3$, then $\mu_F(N_{2,t}) = 0$. Then if $t \notin N_1 \cup N_3$, since E_t is $\mu_F \times m$ measurable we may apply Fubini's theorem to obtain

$$\mu_F \times m(E_t) = \int_F m(E_{(x,t)})\, d\mu_F(x) = 0.$$

Now if $D \subset E$, then $D_t \subset E_t$, and $\mu_F \times m(E_t) = 0$ for a.e. t. Completeness of $\mu_F \times m$ then implies D_t is $\mu_F \times m$ measurable for a.e. t, so $D \in \Sigma_c$.

Now note that if $E \subset \Sigma_r$, then $E^* \in \Sigma_r \otimes \mathscr{M} \subset \Sigma_c$. Hence $E = E_t^* \in \Sigma_F \otimes \mathscr{M}_Q$ for almost every $t \in Q$, proving $\Sigma_r \subset \Sigma_F \otimes \mathscr{M}_Q$.

There is a shorter, alternative proof that $\Sigma_F \otimes \mathscr{M}_Q = \Sigma_r$ using Rokhlin's Theorem on Bases [15]. Recall that $\{M_j\}_1^\infty$ is a separating sequence of measurable sets in X. From the collection $\mathscr{E}$ of finite intersections of the M_j and their complements. By removing an invariant null set, we may assume each set in $\mathscr{E}$ is measurable on every orbit of T. For $E \in \mathscr{E}$ define g_E for $T_u x \in T_Q F$ by

$$g_E(T_u x) = \int_Q \chi_E(T_t x)\, dt.$$

Note that g_E does not depend on u. We claim g_E is μ measurable. Let $Q_m = (1/m)Q$, and define $g_{E,m}$ on $T_{Q_m} F$ by

$$g_{E,m}(T_u x) = \int_Q \chi_E(T_{t+u} x)\, dt \qquad (u \in Q_m,\ x \in F).$$

Measurability of the flow shows that $g_{E,m}$ is measurable on $T_{Q_m} F$. Extend $g_{E,m}$ to all of $T_Q F$ by periodicity. Continuity of $g_{E,m}$ in the flow parameter shows that for all $x \in F$ we have

$$\lim_{m\to\infty} \sup_{u \in Q} | g_{E,m}(T_u x) - g_E(x)| = 0.$$

Hence $g_{E,m} \to g_E$ pointwise, proving measurability of g_E.

Let $\mathscr{D} \subset \Sigma_F$ be the collection of subsets of F of the form

$$\{x \in F\colon a < g_E(t) < b\},$$

where a and b are rational and $E \in \mathscr{E}$. If $\mathscr{D}$ did not separate points of F, there would be an x and y in F for which $g_E(x) = g_E(y)$ for all $E \in \mathscr{E}$. A standard argument shows this would yield an isomorphism U: $T_Q x \to T_Q y$ for which $U(T_Q x \cap E) = T_Q y \cap E$ (at least after removing a null set from each). This means $T_t x$ and $U(T_t x)$ are never separated by $\mathscr{E}$ for almost every $t \in Q$. But this contradicts the point separating property of $\{M_j\}$, and shows $\mathscr{D}$ indeed separates points of F.

If Q_0 is a subrectangle of Q with rational endpoints and $D \in \mathscr{D}$, then $T_Q D \cap T_{Q_0} F = \varphi^{-1}(D \times Q_0)$ is in Σ_r. The countable collection $\mathscr{R}$ of such sets separates points of the Lebesgue space $(T_Q F, \Sigma_r, \mu_r)$, and the measures $\mu_F \times m_Q$ and μ_r agree on $\mathscr{R}$. By Rokhlin's Theorem on Bases

[15], the completion of the σ-algebra generated by $\mathscr{D}$, namely $\Sigma_F \otimes \mathscr{M}_Q$, is all of Σ_r. This completes the proof of the theorem.

3. Aperiodic Decomposition

We will show that if T is an arbitrary n-dimensional flow on (X, Σ, μ), then X decomposes into invariant subsets on each of which T acts aperiodically when factored through a closed subgroup of R^n. These subsets have a natural Lebesgue space structure which is preserved by T, so the general version of Rokhlin's theorem, using quotients of R^n as parameter group, applies to them.

Let T be an n-dimensional flow on X. The construction of the metric in the proof of Rokhlin's theorem did not depend on aperiodicity of the flow, so there is a separable metric d on an invariant conull set, which we may assume is all of X, for which T is continuous. For $x \in X$ define H_x to be $\{t \in R^n: T_t x = x\}$, and let $\mathscr{C}$ denote the class of closed subgroups of R^n. Then continuity of T implies $H_x \in \mathscr{C}$ for all x. Notice that H_x depends only on the orbit of x, i.e., $H_{T_t x} = H_x$ for all t. For $H \in \mathscr{C}$ the set $X_H = \{x \in X: H_x = H\}$ is invariant. The restriction of T to X_H then factors through the quotient map $R^n \to R^n/H$, and the induced R^n/H-flow on X_H is aperiodic. We will prove that the partition $\xi = \{X_H: H \in \mathscr{C}\}$ of X is measurable in the sense of Rokhlin [15]. That is, there is a sequence of measurable sets $\{E_m\}$ such that

$$\left\{\bigcap_{m=1}^{\infty} E_m^{\epsilon_m}: \epsilon_m = \pm 1\right\} = \xi,$$

where $E^{+1} = E$, $E^{-1} = X \backslash E$. This will induce a Lebesgue space structure on each X_H (Rokhlin's "canonical system of measures") and show that X is the direct sum of the X_H as defined by Halmos [4]. This decomposition is the aperiodic analog of Halmos' ergodic decomposition.

Theorem 2. *The partition $\xi = \{X_H: H \in \mathscr{C}\}$ is measurable. Hence there are Lebesgue space measures μ_H on X_H and $\mu_{\mathscr{C}}$ on $\mathscr{C}$ such that if $E \in \Sigma$, then $X_H \cap E$ is μ_H measurable for $\mu_{\mathscr{C}}$ almost every H, $\mu_H(X_H \cap E)$ is a $\mu_{\mathscr{C}}$ measurable function, and*

$$\mu(E) = \int_{\mathscr{C}} \mu_H(X_H \cap E)\, d\mu_{\mathscr{C}}(H).$$

Furthermore, $T \mid X_H$ is a measurable flow preserving the measure μ_H.

Proof. Once ξ is verified to be measurable, the rest follows using the measurable partition machinery of Rokhlin or the results of Halmos.

Let A be a closed set in R^n. We first prove that $\{x: T_t x = x \text{ for some } t \in A\} \in \Sigma$. For fixed t, $d(x, T_t x)$ is a μ measurable function. Let A_0 be a countable dense subset of A. Since $d(x, T_t x)$ is continuous in t, we have $\inf\{d(x, T_t x): t \in A\} = \inf\{d(x, T_t x): t \in A_0\}$. Countability of A_0 shows this is again a μ measurable function. Hence

$$\{x: T_t x = x \text{ for some } t \in A\} = \{x: \inf_{t \in A} d(x, T_t x) = 0\} \in \Sigma.$$

Denote by kB the closed ball in R^n of radius k, and for $H \in \mathscr{C}$ let $H^m = H + \{t \in R^n: |t| < 1/m\}$. For fixed k and m a simple compactness argument applied to $\{H^m \cap kB: H \in \mathscr{C}\}$ shows that there is a finite collection $\mathscr{C}_{km} \subset \mathscr{C}$ such that if $K \in \mathscr{C}$, there is an $H \in \mathscr{C}_{km}$ with

$$K \cap kB \subset H^m \cap kB, \qquad H \cap kB \subset K^m \cap kB. \tag{*}$$

If we let $\mathscr{C}_{kmH} = \{K \in \mathscr{C}: K \cap kB \subset H^m \cap kB\}$, then we claim $\{\mathscr{C}_{kmH}: k, m \geqslant 1, H \in \mathscr{C}_{km}\}$ is a countable point separating class of subsets of $\mathscr{C}$. For if K and K' are different elements of $\mathscr{C}$, choose k large enough so that $K \cap kB \neq K' \cap kB$. By symmetry, we may assume there is a $t_0 \in (K \backslash K') \cap kB$. Choose m such that $2/m < \text{dist}(t_0, K')$. There are $H, H' \in \mathscr{C}_{mk}$ such that (*) holds for K, K' respectively. Hence $\text{dist}(t_0, H) < 1/m$, $\text{dist}(t_0, H') \geqslant \text{dist}(t_0, K') - 1/m > 1/m$. Thus $H \neq H'$ and therefore K and K' are separated by $\{\mathscr{C}_{kmH}\}$.

Let $E_{kmH} = \{x: H_x \in \mathscr{C}_{kmH}\}$. Since $kB \backslash H^m$ is closed, E_{kmH} is measurable because it is the complement of $\{x: T_t x = x \text{ for some } t \in kB \backslash H^m\}$. Measurability of ξ now follows from measurability of the E_{kmH} since $\{E_{kmH}\}$ generates ξ.

Remark. The sets E_{kmH} are invariant under T. Thus if η_T denotes the partition of X into ergodic components under T, then each E_{kmH} is an η_T set. Hence ξ is refined by η_T, that is, $\xi \leqslant \eta_T$. In particular, suppose T is ergodic, so that η_T is trivial. Then there is exactly one closed subgroup H of R^n such that $\mu(X_H) = 1$. In this case the flow is "really" an aperiodic flow on R^n/H. For $n = 1$, this shows ergodicity implies aperiodicity. When $n > 1$ this is no longer strictly true since we must factor T through a quotient group R^n/H to obtain aperiodicity. Thus for n-dimensional flows, if T_t is never the identity for $t \neq 0$, then ergodicity implies aperiodicity.

4. Measurability of Factor Flows

If S is a measure preserving transformation on (X, Σ, μ) and $\mathscr{A}$ is a completely invariant σ-subalgebra of Σ, then S is also measurable (and of course measure preserving) on $(X, \mathscr{A}, \mu)$. However, this does not always happen for flows. For example, let X be the unit circle, with Lebesgue measure μ, $T_t (t \in R)$ rotation by t, Σ the σ-algebra of Lebesgue measurable sets, and $\mathscr{A}$ the countable and cocountable subsets of X. Then $\mathscr{A}$ is completely invariant under $\{T_t\}$, Since any set in $\mathscr{A} \otimes \mathscr{M}$, where $\mathscr{M}$ denotes the Lebesgue measurable subsets of R, depends on only a countable or cocountable set of X coordinates the flow on $(X, \mathscr{A}, \mu)$ is not measurable since, for instance, for each $x_0 \in X$, the set $\{(x, t)\colon T_t x = x_0\}$ is not in $\mathscr{A} \otimes \mathscr{M}$. The basic obstruction to measurability on a factor is the presence of null sets which are transversal in a nonmeasurable way to the flow. The following results shows we can obtain measurability by removing some of the null sets in the σ-subalgebra.

Two σ-subalgebras are *equivalent* if each set in one is equal to a set in the other up to a null set.

THEOREM 3. *Let T be an n-flow on (X, Σ, μ), and suppose that $\mathscr{A}$ is an invariant σ-subalgebra of Σ. Then there is a complete invariant σ-subalgebra $\tilde{\mathscr{A}}$ of $\mathscr{A}$ which is equivalent to $\mathscr{A}$ and such that T is a measurable flow on $(X, \tilde{\mathscr{A}}, \mu)$.*

Proof. Let $\{E_j\}$ be countable and dense in $\mathscr{A}$. Construct a pseudometric d_0 similar to the metric d in the proof of Rokhlin's theorem as follows. Remove an invariant null set in order to assume that $\chi_{E_j}(T_t x)$ is measurable in t for each x. Then the functions

$$g_{jk}(x, t) = \frac{1}{m(B_k)} \int_{B_k} \chi_{E_j}(T_{s+t} x)\, dx \qquad (j, k \geqslant 1)$$

are defined, measurable in x, and continuous in t. Put

$$d_0(x, y) = \sum_{j,k=1}^{\infty} 2^{-j-k} \sup_{t \in B} | g_{jk}(x, t) - g_{jk}(y, t)|.$$

Then d_0 is a pseudometric on X under which the map $t \mapsto T_t x$ is continuous for fixed x.

Form the countable class $\mathscr{E}$ of sets of the form $\{x\colon r \leqslant g_{jk}(x, q) \leqslant s\}$, where $j, k \geqslant 1$, $r < s$ are rational, and q is an n-tuple of rationals. Then $\mathscr{E}$ consists of closed sets since $g_{jk}(x, t)$ is continuous in x for fixed t, and

$\mathscr{E} \subset \mathscr{A}$ since $\mathscr{A}$ is invariant. Let $\mathscr{A}_0$ be the σ-algebra generated by $\{T_t E: E \in \mathscr{E}, t \in R^n\}$. Since $\mathscr{A}_0$ contains a set equivalent to each E_j, it is an invariant σ-subalgebra of $\mathscr{A}$ which is equivalent to $\mathscr{A}$. Forming $\mathscr{A}_0$ has removed the null sets in $\mathscr{A}$ which cause measurability problems.

We claim that if $E = \{x: r \leqslant g_{jk}(x, q) \leqslant s\} \in \mathscr{E}$, then $T^{-1}E = \{(x, t): T_t x \in E\}$ is in $\mathscr{A}_0 \otimes \mathscr{M}$. Let $C_k = [0, 2^{-k})^n$, and $C_k{}^0$ be the rational points in C_k. Note that since $g_{jk}(T_u x, t) = g_{jk}(x, t + u)$, and g_{jk} is continuous in t, we have

$$\begin{aligned} T_{C_k}E &= \{x: r \leqslant g_{jk}(x, q + t) \leqslant s \text{ for some } t \in C_k\} \\ &= \{x: r \leqslant g_{jk}(x, q + t_0) \leqslant s \text{ for some } t_0 \in C_k{}^0\} \\ &= \bigcup_{t_0 \in C_k{}^0} \{x: r \leqslant g_{jk}(x, q + t_0) \leqslant s\}, \end{aligned}$$

which shows that $T_{C_k}E \in \mathscr{A}_0$. Let $C_{km} = 2^{-k}m + C_k$, where $m \in Z^n$, and put

$$F_k = \bigcup_{m \in Z^n} T_{C_{km}}E \times C_{km},$$

which is a set in $\mathscr{A}_0 \times \mathscr{M}$. Clearly $T^{-1}E \subset F_k$ for each k, and $\{F_k\}$ decreases. If $(x, t) \in \bigcap F_k$, then for each k there is an t_k with $| t_k | < 2^{-k}$ such that $T_{t_k} x \in T_t E$. Because $T_t E$ is closed, letting $k \to \infty$ shows that $x \in T_t E$. Hence $(x, t) \in T^{-1}E$, and thus $F_k \searrow T^{-1}E$. This proves $T^{-1}E \in \mathscr{A}_0 \times \mathscr{M}$.

Since $T^{-1}\mathscr{E} \subset \mathscr{A}_0 \times \mathscr{M}$, it follows that $T^{-1}\mathscr{A}_0 \subset \mathscr{A}_0 \times \mathscr{M}$. Let $\tilde{\mathscr{A}}$ be the completion of $\mathscr{A}_0$ with respect to μ. If $N \in \tilde{\mathscr{A}}$ with $\mu(N) = 0$, then there is an $N_0 \in \mathscr{A}_0$ such that $N \subset N_0$ and $\mu(N_0) = 0$. Since $T^{-1}N \subset T^{-1}N_0 \in \mathscr{A}_0 \times \mathscr{M}$ and $\mu \times m(T^{-1}N_0) = 0$, we have

$$T^{-1}N \in \mathscr{A}_0 \otimes \mathscr{M} \subset \tilde{\mathscr{A}} \otimes \mathscr{M}.$$

This shows $\tilde{\mathscr{A}}$ satisfies the conclusions of the theorem, and completes the proof.

Remark. This result shows that Rokhlin's theorem holds for nonatomic factors of aperiodic flows as well. For if $\mathscr{A}$ is a nonatomic factor of T, the above says that by removing some null sets we may assume T is measurable on $(X, \mathscr{A}, \mu)$. If ξ denotes the measurable partition of X induced by $\mathscr{A}$, then T defines an aperiodic measurable flow on the nonatomic factor Lebesgue space $(X/\xi, \mathscr{A}_\xi, \mu_\xi)$. If F is the set produced by Rokhlin's theorem applied to this flow, then the inverse image of F under the quotient map $X \to X/\xi$ satisfies the conclusions of Theorem 1.

5. Factors of Finite Entropy

The restriction of an n-flow T to Z^n defines an abelian group of measure preserving transformations. Conze [3] and Katznelson and Weiss [6] have defined the entropy of such a group on the following way. The entropy of a partition α is denoted by $H(\alpha)$. If $H(\alpha) < \infty$, then $\lim_{Q \to \infty} | Q |^{-1} H(\bigvee_{m \in Q} T_m \alpha)$ exists, where "$Q \to \infty$" means rectangles in Z^n whose sides become arbitrarily large. This limit is denoted by $h(T \mid Z^n, \alpha)$. We define $h(T \mid Z^n)$ to be $\sup\{h(T \mid Z^n, \alpha): H(\alpha) < \infty\}$, and the entropy $h(T)$ of the flow to be the entropy $h(T \mid Z^n)$ of the discretized flow. A basic theorem in the subject is that if α generates under $T \mid Z^n$, then $h(T \mid Z^n) = h(T \mid Z^n, \alpha)$ [3, p. 18].

It is easy to produce factors of finite entropy for the discretized flow. For example, the entropy of $T \mid Z^n$ on the invariant σ-subalgebra generated by α is bounded by $H(\alpha)$. However, this σ-algebra may not be invariant under the entire flow, and obtaining a factor of finite entropy for the flow is not so simple. The following theorem shows that this can always be done.

Theorem 4. *If T is an n-flow on X, there is a nontrivial partition α of X with finite entropy such that $\bigvee \{T_t \alpha: t \in R^n\} = \bigvee \{T_m \alpha: m \in Z^n\}$. Thus T has a factor of finite entropy on which it is measurable.*

Proof. The second statement follows easily from the first. For if $\mathscr{A}$ denotes the σ-algebra corresponding to the partition $\bigvee \{T_t \alpha: t \in R^n\}$, then $\mathscr{A}$ is invariant under T. Theorem 3 shows that we may assume T is measurable on $(X, \mathscr{A}, \mu)$. Since α generates $\mathscr{A}$ under $T \mid Z^n$, statements from the first paragraph show that

$$h(T \mid Z^n \text{ on } \mathscr{A}) = h(T \mid Z^n, \alpha) \leqslant H(\alpha) < \infty.$$

We may aslo assume T is aperiodic. For the following proof shows $H(\alpha)$ can be bounded by a constant M independent of T. If ξ denotes the aperiodic decomposition of X from Theorem 2, then by working on the fibers of ξ we can construct α so that the first statement of the theorem holds on fibers of ξ. Then $H(\alpha) \leqslant \int_{X/\xi} H(\alpha_C)\, d\mu_\xi(C) \leqslant M < \infty$, where α_C is the restriction of α to $C \in \xi$, and μ_ξ is the quotient measure on X/ξ.

We use the idea of the name of a point with respect to a flow and a partition. If α is a measurable partition of X and T is an n-flow, for each $x \in X$ define $A_x: R^n \to \alpha$ by $T_t x \in A_x(t) \in \alpha$. For almost every x the function A_x is measurable, and is called the *continuous* α-*name of* x. The restriction $A_x \mid Z^n$ of A_x to Z^n is termed the *discrete* α-*name of* x. The idea of the proof, due to Ornstein, is to construct partitions α^k converging to α for which the discrete α^k-name of a point determines its continuous α-name with increasing accuracy. In the limit, the discrete α-name of a point will exactly determine its continuous α-name. Hence two points separated by $\{T_t\alpha: t \in R^n\}$ must have already been separated by $\{T_m\alpha: m \in Z^n\}$, that is $\bigvee \{T_t\alpha: t \in R^n\} = \bigvee \{T_m\alpha: m \in Z^n\}$.

Choose $\epsilon_k \searrow 0$ such that $\sum_1^\infty \epsilon_k < \infty$, and positive integers $m_k \nearrow \infty$ such that

$$\sum_{k=1}^{\infty} \frac{1}{m_k{}^n \epsilon_k{}^n} < \infty, \qquad -\sum_{k=1}^{\infty} \frac{10^n}{m_k{}^n \epsilon_k{}^n} \log \frac{10^n}{m_k{}^n \epsilon_k{}^n} < \infty, \qquad \frac{30}{\epsilon_k} < m_k .$$

Let $Q_k = [0, m_k)^n$. We construct inductively partitions α^k converging to the desired partition α.

Construct α^1 as follows. Use the Rokhlin theorem to find a Q_1-set F_1 with $\mu(T_{Q_1}F_1) > 1 - \epsilon_1$. Choose numbers t_i, $1 \leqslant i \leqslant r = [\epsilon_1^{-1}]$, such that $t_r < 30\epsilon_1^{-1}$, $20 \leqslant t_{i+1} - t_i < 30$, $t_1 = 0$, and the fractional parts of the t_i are ϵ_1 dense in $[0, 1)$. Let

$$A_1 = \bigcup \{T_t F_1 \colon t \in [t_{i_1}, t_{i_1} + 10) \times \cdots \times [t_{i_n}, t_{i_n} + 10),\ 1 \leqslant i_1, \ldots, i_n \leqslant r\},$$

$B = X \backslash A_1$, and $\alpha^1 = \{A_1, B\}$. Define a 1-block in the continuous α^1-name of x to be the restriction of A_x to $t_0 + Q_1$, where $T_{t_0}x \in F_1$. The 1-blocks are uniquely determined by the continuous α^1-name, and they are determined up to a translate of at most ϵ_1 by the discrete α^1-name.

Continue the construction to α^k as follows. We already have produced $\alpha^{k-1} = \{A_1, \ldots, A_{k-1}, B\}$, where the continuous α^{k-1}-name of a point breaks up into $(k-1)$-blocks which are determined by the discrete α^{k-1}-name of the point up to a translate of at most ϵ_{k-1}. Find a Q_k-set F_k with $\mu(T_{Q_k}F_k) > 1 - \epsilon_k$. We can measurably modify α^{k-1} to assume that each $(k-1)$-block in the α^{k-1}-name of points in F_k begins at a multiple of $2\epsilon_{k-1}$. Put all of the $(k-1)$-blocks which intersect $T_{[0,30\epsilon_k^{-1})^n}F_k$ into B. Pick $[\epsilon_k^{-1}]$ numbers t_i such that $20 \leqslant t_{i+1} - t_i \leqslant 30$, $0 \leqslant t_i \leqslant 30\epsilon_k^{-1}$, $t_1 = 0$, and the fractional parts of the t_i are ϵ_k dense in $[0, 1)$. Define

$$A_k = \bigcup \{T_t F_k \colon t \in [t_{i_1}, t_{i_1} + 10) \times \cdots \times [t_{i_n}, t_{i_n} + 10),\ 1 \leqslant i_1, \ldots, i_n \leqslant \epsilon_k^{-1}\},$$

and let $\alpha^k = \{A_1, \ldots, A_k, B\}$, where $B = X \backslash (A_1 \cup \cdots \cup A_k)$.

The continuous α^k-name of a point breaks up into k-blocks beginning with a point in F_k. Also, the discrete α^k-name determines the A_k portion of the k-block up to a translate of at most ϵ_k, and since the $(k-1)$-blocks are determined within an error of ϵ_{k-1} and occur at multiples of $2\epsilon_{k-1}$, they are exactly determined inside the k-block. Thus, the discrete α^k-name determines k-blocks to within ϵ_k.

The change in partition distance at the kth state is estimated by

$$| \alpha^k - \alpha^{k-1} | < \frac{2^n}{10} \epsilon_{k-1} + \frac{30^n}{\epsilon_k{}^n m_k{}^n} + \frac{10^n}{\epsilon_k{}^n m_k{}^n},$$

which is summable by our choice of ϵ_k and m_k. Hence the α_k converge in partition distance to some partition α. Since

$$-\mu(A_k) \log \mu(A_k) < -\frac{10^n}{\epsilon_k{}^n m_k{}^n} \log \frac{10^n}{\epsilon_k{}^n m_k{}^n},$$

α has finite entropy. Finally, we claim that the discrete α-name of a point determines its continuous α-name. For the continuous α-name breaks up into k-blocks. Each k-block is contained in a $(k+r)$-block for all large r. This $(k+r)$-block is determined by the discrete α-name to within ϵ_{k+r}, so the original k-block is determined by the discrete α-name to within ϵ_{k+r}. Since this is arbitrarily small the discrete α-name determines k-blocks exactly, hence the continuous α-name. This completes the proof of the theorem.

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