FINITARILY SPLITTING SKEW PRODUCTS

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§1. Introduction

Skew products with ergodic automorphisms of compact abelian groups arise naturally in several contexts. For example, suppose S is an automorphism of the compact group G, and H is an S-invariant closed subgroup. By taking a measurable cross section to the quotient map $G \to G/H$, the transformation S can be regarded as a skew product of the quotient automorphism $S_{G/H}$ with the restriction S_H of S to H. We can study S by studying the simpler components, $S_{G/H}$ and S_H, and how they are joined in a skew product. This method was used in proving that ergodic automorphisms of compact groups are measure theoretically isomorphic to Bernoulli shifts [3]. Crucial to this method is the result that if S_H is ergodic, then the skew product S measure theoretically splits into the direct product $S_{G/H} \times S_H$.

In this example, however, there is additional structure. The base map $S_{G/H}$ is continuous, and the cross section to the quotient map can be chosen to be almost continuous (or finitary), i.e., continuous off a meager null set (Theorem 2). It is natural to ask whether the isomophism of S with $S_{G/H} \times S_H$ can also be made almost continuous. We show that the answer is "yes," and prove a general almost continuous splitting theorem assuming a mild condition on the base map.

One of our motivations is the search for "natural" examples of transformations that are measure theoretically but not almost continuously isomorphic to Bernoulli shifts, i.e., that are Bernoulli but not "finitarily Bernoulli." Ergodic toral automorphisms with off-diagonal 1's in Jordan blocks of eigenvalues of modulus one (called central skew automorphisms) do not obey weak specification [5], and fine enough smooth

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*Supported in part by NSF Grant MCS77-04915

partitions are never weak Bernoulli. Hence they are natural candidates for such examples. However, our work shows that a central skew automorphism is almost continuously isomorphic to a finite factor of a diagonalizable automorphism. Techniques developed by Rudolph [6] should be sufficient to show the latter to be finitarily Bernoulli, and thus prove that all ergodic toral automorphisms are finitarily Bernoulli.

Skew products are closely related to cocycles, and the splitting of skew products can be cast in the form of "straightening out" certain cocycles with values in the affine group of G. In §3 we formulate this precisely, and show how this suggests analogous results about skew product actions of groups more complicated than the integers.

In §4 we give the modifications in the specification argument of [4] needed to get finitary splitting, and in §5 are the necessary lemmas to go from automorphisms with specification to general ergodic automorphisms.

In §6 we show that if the group automorphism is "hyperbolic" in a certain general sense, then skew products with it finitarily split with no conditions on the base map. The proof uses a Neumann series argument shown to us in the toral case by W. Parry.

The author wishes to give his hearty thanks to the organizers of the Special Year in Ergodic Theory at the University of Maryland and to the Workshop on Ergodic Theory, Institute for Advanced Studies, Hebrew University, Jerusalem, July-August 1980, for their support while this paper was being written.

§2. Finitary splitting

We first set up the general framework to state the theorem. Let (X,d) be a compact metric space, and μ be a nonatonic Borel probability measure on X. By throwing out the largest open μ-null set, we can and will assume that μ is strictly positive on nonempty open sets. In particular, if $B(x,\varepsilon)$ denotes the ball of radius $\varepsilon > 0$ centered on $x \in X$, then $\mu(B(x,\varepsilon)) > 0$.

An almost _continuous_ (or _finitary_) automorphism of (X,μ)

is given by a pair (X_1, U), where $X \setminus X_1$ is meager and null, $U\colon X_1 \to X_1$ is a homeomorphism in the induced topology on X_1, and U preserves the trace of μ on X_1. In this case we will call U a <u>map</u> of X, it being understood that U is defined only up to an invariant meager null set. Two such maps are <u>finitarily isomorphic</u> if by removing further meager null sets there is measure-preserving homeomorphism conjugating them. For further discussion of maps, see Denker and Keane [1]. There they show that there exists a totally bounded metric d_1 on X_1 equivalent to the original metric d such that U is uniformly continuous on (X_1, d_1). Hence U extends to the compact completion of (X_1, d_1). Therefore we can and will think of U on X_1 as the restriction of a homeomorphism of a compact metric space (X, d_1) to an invariant residual set X_1 of full measure.

From now on we will assume that U is ergodic on (X, μ).

Let G be a metrizable compact abelian group (hereafter abbreviated "compact group") and S be a continuous, algebraic automorphism of G. Then S preserves Haar measure m on G. If $\alpha\colon X \to G$ is measurable, the skew product transformation $U \times_\alpha S\colon X \times G \to X \times G$ defined by

$$(U \times_\alpha S)(x, g) = (Ux, Sg + \alpha(x))$$

preserves $\mu \times m$.

We showed in [4] that if S is ergodic, then $U \times_\alpha S$ is isomorphic to the direct product $U \times S$ via an isomorphism of the form $W(x, g) = (x, g + \beta(x))$, where $\beta\colon X \to G$ is measurable. This amounts to solving the functional equation

$$\alpha(x) = \beta(Ux) - S\beta(x) \tag{1}$$

for β, given α, U, and S.

Suppose now that $\alpha\colon X \to G$ is almost continuous, i.e. continuous after removing a meager null set from X. Then $U \times_\alpha S$ is a map, and it is finitarily isomorphic to $U \times X$ if there is an almost continuous solution β to (1). We will show that such a β exists if U satisfies the following condition.

Definition: A map U of (X,d) is compressible if for every positive integer n and every $\varepsilon > 0$, there is an aperiodic $x \in X$ such that $\operatorname{diam} U^j x:\{0 \le j \le n\} < \varepsilon$.

The purpose of compressibility is to guarantee that modifications on long pieces of U orbit during the construction of β can be made topologically as well as measure theoretically small.

Theorem 1: Suppose U is compressible and S is ergodic. Then for each almost continuous skewing function $\alpha:X \to G$ there exists an almost continuous solution β to the functional equation (1). Hence the finitary skew product $U \times_\alpha S$ is always finitarily isomorphic to the direct product $U \times S$.

Note that if U has a fixed point x_0, which is a limit of aperiodic points, then small perturbations of x_0 yield long pieces of orbit with small diameter, and so U is compressible. Since automorphisms fix the identity, Theorem 1 applies to the situation described in §1 to show that S is finitarily isomorphic to $S_{G/H} \times S_H$.

§3. Cohomological interpretation

The measurable splitting of skew products in [4] can be interpreted as "straightening out" certain kinds of cocycles. Recast in this form, the result is similar to Zimmer's rigidity theorem for ergodic actions of semi-simple Lie groups [7]. A beautiful exposition of Zimmer's work has recently been given by Furstenberg [2]. This interpretation suggests a suitable framework for questions involving skew product actions of groups more complicated than the integers $\mathbb{Z}$.

The transformation U gives a measure-preserving action of $\mathbb{Z}$ on (X,μ). The automorphism S induces a homomorphism π from $\mathbb{Z}$ to the automorphism group of G by $\pi(n) = S^n$. This defines the semi-direct product group $\mathbb{Z} \times_\pi G$ with multiplication $(n_1,g_1)\cdot(n_2,g_2) = (n_1+n_2,g_1+S^{n_1}g_2)$. This semi-direct product acts affinely on G by $(n,g)\cdot g' = S^n g' + g$. The skew product $U \times_\alpha S$ yields a cocycle $\sigma\colon \mathbb{Z} \times X \to \mathbb{Z} \times_\pi G$

defined by

$$(U \times_\alpha S)^n(x,g) = (U^n x, \sigma(n,x)\cdot g).$$

This σ clearly obeys the cocycle equation

$$\sigma(n_1+n_2,x) = \sigma(n_1, U^{n_2}x)\sigma(n_2,x).$$

More explicitly, $\sigma(n,x) = (n, \alpha_n(x))$, where

$$\alpha_n(x) = \alpha(U^{n-1}x) + S\alpha(U^{n-2}x) + \cdots + S^{n-1}\alpha(x)$$

for $n \geq 1$, and a similar formula for $n \leq -1$. Thus σ is "level-preserving" in the sense that $\sigma(\{n\} \times X) \subset \{n\} \times G$, and every level-preserving cocycle corresponds to a skew product $U \times_\alpha S$.

The direct product $U \times S$ corresponds to the trivial cocycle $\tau(n,x) = (n,0)$. If $\beta\colon X \to G$ solves (1), put $\psi(x) = (0,\beta(x)) \in \mathbb{Z} \times_\pi G$. An easy computation shows that $\alpha_n(x) = \beta(U^n x) - S^n\beta(x)$, and therefore that

$$\psi(U^n x)^{-1}\sigma(n,x)\psi(x) = \tau(n,x). \tag{2}$$

Thus σ is cohomologous to τ via the coboundary defined by ψ, i.e., σ can be "straightened out."

We remark that proving σ is cohomologous to the trivial cocycle is equivalent to solving (1). For suppose $\psi\colon X \to \mathbb{Z} \times_\pi G$ obeys (2). If $\psi(x) = (p(x),\beta(x))$, then for $n = 1$ the first coordinates of (2) yield

$$-p(Ux) + 1 + p(x) = 1,$$

so $p(x)$ is U-invariant. Since we assume U to be ergodic, $p(x) = n_0$ almost everywhere. Applying S^{-n_0} to the second coordinates shows that β solves (1).

The splitting theorem of [4] therefore says exactly that every level-preserving cocycle $\sigma\colon \mathbb{Z} \times X \to \mathbb{Z} \times_\pi G$ is cohomologous to one that is independent of X. Zimmer's work [7] shows that every cocycle $\sigma: H \times X \to K$, where H is a suitable Lie group acting ergodically on (X,μ) preserving μ and K is another suitable Lie group, is cohomologous to one that is independent of X. The results are similar, but the groups operating are quite different.

This suggests the following general question. Let Γ be a countable discrete group acting ergodically on (X,μ), and π be a homomorphism from Γ to the automorphism group of G. The semi-direct product $\Gamma \times_\pi G$ acts affinely on G. When is a level-preserving cocycle $\sigma: \Gamma \times X \to \Gamma \times_\pi G$ cohomologous to the trivial cocycle $\tau(\gamma,x) = (\gamma,0)$? Equivalently, when does the corresponding skew product action of Γ on $X \times G$ split into the direct product action? When $\Gamma = \mathbb{Z}^n$, G is a torus, and $\pi(\gamma)$ is hyperbolic for $\gamma \neq e$ we have shown all cocycles are trivial. However, results available now are quite fragmentary.

§4. Automorphisms with weak specification

If S obeys weak specification, the measurable solution β to (1) found in [4, §4] is not necessarily almost continuous for the following reason. Uncontrolled modifications produced by using weak specification at each stage of the construction of β occur in long gaps between Rohlin stacks in X. Although these gaps become measure theoretically negligible, so the approximations converge a.e., they can also become topologically dense. In fact, this attempted argument for almost continuity of β breaks down when U is an irrational rotation of the circle. For this U the almost continuous solvability of (1) is still in doubt. This is the reason to impose compressibility on U.

First we establish what we need of compressible transformations. We denote the closure of a subset E by $cl(E)$. A subset F of X is almost open if F and $X \setminus F$ agree with open sets up to a meager null set. This means that the "essential boundary" of F has measure zero. It is easy to see that for fixed $x \in X$, the ball $B(x,\delta)$ is almost open for all but countably many values of δ.

Lemma 1: Suppose U is a compressible map of (X,d,μ). Let $\{n_j\}$ and $\{m_j\}$ be increasing sequences of natural numbers, and $\varepsilon_j \searrow 0$. Then there are almost open sets $F_j \subset X$ of positive measure such that

(i) $\{U^i F_j: -n_j \le i < m_j\}$ is a disjoint collection for each j,

(ii) if $L_k = \cup \{U^i F_k: -n_j \le i < 0\}$, then

$$\operatorname{diam}[\operatorname{cl}(\bigcup_{k \in j} L_k)] < \varepsilon_j.$$

Proof: By ergodicity and compressibility of U, there are $x_j \in X$ with infinite U-orbit such that

$$\operatorname{diam}\{U^i x_j: -n_j \le i < 0\} < \varepsilon_j/8.$$

Since (X,d) is assumed compact, by taking a convergent subsequence we can assume that there is an $x_0 \in X$ such that $d(x_j,x_0) < \varepsilon_j/8$. Continuity of U shows there are $\delta_j > 0$, $\delta_j < \varepsilon_j/8$ such that if $F_j = B(x_j,\delta_j)$, then F_j is almost open and $\{U^i F_j: -n_j \le i < m_j\}$ is disjoint. Also, $\mu(F_j) > 0$ since nonempty open sets have positive measure, which proves (i). Finally, $\cup\{L_k: k \ge j\} \subset B/x_0, 3\varepsilon_j/8)$, proving (ii).

Next we recall the weak specification property.

Definition: A homeomorphism f of a compact metric space (Y,d) satisfies weak specification if for every $\varepsilon > 0$ there is an integer $M(\varepsilon)$ such that for every $r \ge 2$ and r points $y_1,\ldots,y_r$ in Y, and for every set of integers $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_r \le b_r$ with $a_j - b_{j-1} \ge M(\varepsilon)$ $(2 \le j \le r)$, there is a $y \in Y$ with $d(f^i y, f^i y_j) \le \varepsilon$ for $a_j \le i \le b_j$, $1 \le j \le r$.

For further details about this property, see [4]. We show there that certain basic group automorphisms have weak specification. On the other hand, not all ergodic group automorphisms have this property [5], answering a question raised in [4].

For those that do, [4] gives a simple proof of skew splitness.

We indicate here the modifications in the proof of Theorem 4.2 of [4] needed to obtain an almost continuous solution β to (1), i.e., to show that the skew product finitarily splits.

Proposition: If U is compressible and S obeys weak specification, then $U \times_\alpha S$ finitarily splits for each almost continuous skewing function α.

Proof: Two changes in the proof in [4, §4] are needed. The first is to replace Rohlin towers with Kakutani skyscrapers to control gap size. The second is to use compressibility of U to force uncontrolled specification adjustments on pieces of orbit into a topologically small set.

Suppose $\alpha: X \to G$ is almost continuous. Choose $\varepsilon_j \searrow 0$ with $\sum \varepsilon_j < \infty$. Let $M(\varepsilon)$ be the number determined by weak specification of S. For $n_j = M(\varepsilon_j)$ and $m_j = M(\varepsilon_j)/\varepsilon_j$, choose almost open sets F_j in accordance with Lemma 1.

For $x \in F_j$ define

$$h_j(x) = \min\{n: n > 0,\ U^n x \in F_j\} - n_j.$$

Since F_j is almost open, h_j is almost continuous on F_j. Put $E_j = \cup\{U^i x: x \in F_j,\ 0 \le i < h_j(x)\} = X \backslash L_j$.

Now define $\beta_1: F_1 \to G$ arbitrarily but almost continuously. As in [4], β_1 extends to an almost continuous function on E_1 satisfying (1) where defined.

If ρ is a translation invariant metric on G, and $f: E_k \to G$, put $\|f\|_{E_k} = \sup\{\rho(0,f(x)): x \in E_k\}$.

We shall construct $\beta_2: E_2 \to G$, from which the inductive step for defining β_k will be clear. The decomposition of F_2 into subsets K consisting of those points in F_2 with the same return time to F_2 and the same entry times $a_1,\dots,a_r$ into F_1 before returning to F_2 is an almost open partition. Thus it is enough to define β_2 over such sets K.

Begin by defining β_2' to be constant on K, and using

(1) to extend β_2' to $\cup\{U^iK: 0 \le i < h_2(K)\}$, where $h_2(K)$ is the constant value of h_2 on K. Now β_1 is already defined on the blocks.

$$\cup\{U^iK: a_j \le i < b_j = a_j + h_1(U^{a_j}K)\}, \qquad 1 \le j \le r,$$

where $a_{j+1} - b_j = M(\varepsilon_1)$. Note the use of skyscrapers to control gap size. The calculation in [4] shows that the error $\beta_2' - \beta_1$ on $\{U^ix: a_j \le i < b_j\}$ is the orbit of a point. By weak specification, there is an adjustment $\beta_2(x)$ of $\beta_2'(x)$ such that this error is uniformly less than ε_1 for $1 \le j \le r$. Since only finite conditions are involved for each K, this adjustment can be made almost continuously. The new function β_2 has the properties that it is almost continuous on E_2, it solves (1) where defined, and $\|\beta_2-\beta_1\|_{E_1} < \varepsilon_1$.

Inductively we obtain almost continuous functions $\beta_k: E_k \to G$ satisfying (1) where defined, and with $\|\beta_{k+1}-\beta_k\|_{E_k} < \varepsilon_k$. Hence $\{\beta_k\}$ converges uniformly off $\mathrm{cl}(\cup_{k\ge j}L_k)$. The latter sequence of sets nests down to a point. Hence $\beta = \lim \beta_k$ is almost continuous and satisfies (1) a.e.

§5. Extension to general group automorphisms

Not all ergodic group automorphisms obey weak specification (e.g. the central skew automorphisms mentioned in §1). However, each is built up from two basic kinds, namely irreducible solenoidal automorphisms and group shifts, by the processes of products, factors, inverse limits, and skew products with basic automorphisms. These basic automorphisms obey weak specifications, hence finitarily split over compressible base maps.

The proof in [4] shows that measurable splitting is preserved under products, factors, skew products, and the kinds of inverse limits encountered in constructing general automorphisms. We give here the facts needed to extend this proof to finitary splitting, and therefore to prove Theorem 1. The extra ingredient is finding an almost continuous cross section to a group quotient map. Bord cross sections are well-known, but we have been unable to find this result in the literature.

Theorem 2: Let G be a compact abelian group, H be a closed subgroup of G, and $\pi: G \to G/H$ be the quotient map. Then there is an almost continuous map $\theta: G/H \to G$ such that $\pi\theta$ is the identity.

Proof: Let G have countable discrete dual group Γ, and G/H have dual $\Delta \subset \Gamma$. Suppose $g \in G/H$, so $g: \Delta \to \mathbb{T}$ is a character. We will give a recipe to extend g to $\tilde{g}: \Gamma \to \mathbb{T}$, and then define $\theta(g) = \tilde{g}$. Since $\tilde{g}|_\Delta = g$, θ will be a cross section to π. We will show that there is a meager null set $E \subset G/H$ such that θ is continuous off E.

Choose $\{\gamma_k: k \geq 1\} \subset \Gamma$ that together with Δ generate Γ. Let Δ_n be a subgroup of Γ generated by Δ and $\gamma_1, \ldots, \gamma_n$, and put $\Delta_0 = \Delta$. We will extend g successively to each Δ_n, and hence yield an extension to $\Gamma = \cup\Delta_n$.

Define $\tilde{g}_0 = g$, and let $n \geq 1$. Suppose g has been extended to $\tilde{g}_{n-1}: \Delta_{n-1} \to \mathbb{T}$, consistently in the sense that $\tilde{g}_{n-1}|_{\Delta_k} = \tilde{g}_k$ $(0 \leq k \leq n-1)$. We will construct $\tilde{g}_n: \Delta_n \to \mathbb{T}$.

Consider γ_n. If there is an integer k such that $k\gamma_n \in \Delta_{n-1}$, defined k_n to be the least positive such integer. Otherwise define $k_n = \infty$. Thus k_n is the order of $\gamma_n + \Delta_{n-1}$ in Γ/Δ_{n-1}.

For $k = 1,2,\ldots,$ define a function $r_k: \mathbb{T} \to \mathbb{T}$ as follows. Let $t_0 = \exp(-2\pi i\xi_0)$ where ξ_0 is an irrational number within $\frac{1}{100}$ of $\frac{1}{2}$. For $t = \exp(2\pi i\theta)$ with $-\xi_0 \leq \theta < -\xi_0 + 1$, define $r_k(t) = \exp(2\pi i\theta/k)$. Thus $r_k(t)^k = t$, and r_k is continuous except at t_0, which is close to -1.

If $k_n < \infty$, define $\tilde{g}_n(\gamma_n) = r_{k_n}(\tilde{g}_{n-1}(k_n\gamma_n))$. Since $\gamma_n + \Delta_{n-1}$ generates Δ_n/Δ_{n-1}, this defines an extension $\tilde{g}_n$ of $\tilde{g}_{n-1}$ to Δ_n. Consistency follows because

$$\tilde{g}_n(k_n\gamma_n) = \tilde{g}_n(\gamma_n)^{k_n} = r_{k_n}(\tilde{g}_{n-1}(k_n\gamma_n))^{k_n} = \tilde{g}_{n-1}(k_n\gamma_n),$$

and $\mathbb{Z}\gamma_n \cap \Delta_{n-1}$ is generated by $k_n\gamma_n$.

If $k_n = \infty$, define $\tilde{g}_n(\gamma_n) = 1$. In this case $\mathbb{Z}\gamma_n \cap \Delta_{n-1} = 0$, so $\tilde{g}_n$ is automatically consistent.

This sequence $\{\tilde{g}_n\}$ of extensions converges to a limit $\tilde{g} = \theta(g): \Gamma \to \mathbb{T}$ such that $\tilde{g}|_{\Delta_n} = \tilde{g}_n$. Of course θ depends on our choice of γ_n and ξ_0.

A discontinuity of θ on G/H can be introduced at stage n only if $k_n < \infty$ and $\tilde{g}_{n-1}(k_n\gamma_n) = t_0$, the point of discontinuity of r_{k_n}. We will show that (a) the set E_n of $g \in G/H$ such that $\tilde{g}_{n-1}(k_n\gamma_n) = t_0$ is contained in a coset of a nowhere dense null subgroup of G, and (b) off $\bigcup_{n\geq 1} E_n = E$ the map θ is continuous.

(a) If $k_n < \infty$, let $a(n) = k_n$. If $k_n = \infty$, put $a(n) = 1$. Let $A(n) = a(n)a(n-1)\cdots a(1)$. If $k_n < \infty$, let $q_n: \Delta_n \to \Delta_{n-1}$ be multiplication by k_n. If $k_n = \infty$, then $\Delta_n = \Delta_{n-1} \oplus \mathbb{Z}\gamma_n$, and let $q_n: \Delta_n \to \Delta_{n-1}$ be projection to the first coordinate.

Suppose now $k_n < \infty$ and $\tilde{g}_{n-1}(k_n\gamma_n) = t_0$. Then

$$\begin{aligned} t_0^{A(n-1)} &= \tilde{g}_{n-1}(k_n\gamma_n)^{A(n-1)} \\ &= \tilde{g}_{n-1}(q_n\gamma_n)^{a(n-1)A(n-2)} \\ &= \tilde{g}_{n-2}(q_{n-1}q_n\gamma_n)^{A(n-2)} \\ &= \cdots \\ &= \tilde{g}_0(q_1q_2\cdots q_n\gamma_n), \end{aligned}$$

i.e. $g(q_1q_2\cdots q_n\gamma_n) = \exp(2\pi i A(n-1)\xi_0)$. Since $q_1q_2\cdots q_n\gamma_n \in \Delta$, we only need the following lemma, with $\xi_1 = A(n-1)\xi_0$, $\gamma = q_1q_2\cdots q_n\gamma_n$, and G replacing G/H.

<u>Lemma 2</u>: Let G be a compact abelian group with dual Γ, let $\gamma \in \Gamma$, and $t_1 = \exp(2\pi i\xi_1)$ where ξ_1 is irrational.

Then $E = \{g \in G: g(\gamma) = t_0\}$ is either empty or a coset of a closed nowhere dense null subgroup of G.

Proof: If γ has finite order, then $g(\gamma)$ is a root of unity, and hence $E = \emptyset$.

Suppose γ has infinite order. The mapping $g \to g(\gamma)$ is then a homomorphism from G onto $\mathbb{T}$, whose kernel K is therefore a closed nowhere dense null subgroup of G. Clearly E is a coset of K, completing the proof of the lemma.

(b) Suppose $g \notin E$. A small change in g produces a small change in $\tilde{g}_1$, hence in $\tilde{g}_2$, etc. If we define $\theta_n(g) = \tilde{g}_n$, then $\theta_n: G/H \to G/\Delta_n^{\perp}$ is continuous off E. Since G is the inverse limit of the $G/\Delta_n^{\perp}$, the limit θ of $\{\theta_n\}$ is also continuous off E.

Corollary: If $\alpha: X \to G/H$ is almost continuous, then there is an almost continuous lifting $\tilde{\alpha}: X \to G$ of α such that $\pi\tilde{\alpha} = \alpha$.

Proof: Let $\theta: G/H \to G$ be a cross section to π continuous off a meager null set E. Some care is needed, since $\alpha^{-1}(E)$ could have positive measure in X and then $\theta\alpha$ need not be almost continuous.

For fixed $z \in G/H$ let $\theta_z(y) = \theta(y-z) + \theta(z)$. Then θ_z is again a cross section to π, continuous off $E + z$. Define the measure $\alpha(\mu)$ on G/H by $\alpha(\mu)(F) = \mu(\alpha^{-1}F)$. By Fubini's theorem,

$$\int_{G/H} \alpha(\mu)(E+z)dm(z) = \int_{G/H} m(E-z)d\alpha(\mu)(z) = 0,$$

so $\alpha(\mu)(E+z) = 0$ for m-almost every $z \in G/H$. For such z, we have $\mu(\alpha^{-1}(E+z)) = 0$, and it follows that $\alpha^{-1}(E+z)$ is also meager since μ is positive on open sets. Thus $\tilde{\alpha} = \theta_z\alpha$ works, finishing the proof.

For the remainder of this section, say that S finitarily splits if for each compressible U and almost continuous α, there is an almost continuous solution β of (1). Using the Corollary to find almost continuous instead of measurable liftings of α, the proofs of the following lemmas are easy adaptations of those in [4, §5]. Together with the structure of general group automorphisms given in [4, §7], they are sufficient to show that every ergodic group automorphism finitarily splits, i.e. to prove Theorem 1.

Lemma 3: Let S be an automorphism of G, and H be an S-invariant closed subgroup of G. If S finitarily splits, so does $S_{G/H}$.

Lemma 4: If S_H and $S_{G/H}$ finitarily split, then so does S, a solution in G/H lift to solutions in G.

The meaning of the last statement is that if $\alpha: X \to G$ and $\beta_1: X \to G/H$ solves (1) for $\pi\alpha$, then there is a solution $\beta: X \to G$ for α such that $\pi\beta = \beta_1$.

Lemma 5: Suppose S is an automorphism of G, and that $H_k \searrow 0$ are S-invariant subgroups with $H_0 = G$. If S_{H_{k-1}/H_k} finitarily splits for $k \geq 1$, then S finitarily splits.

§6. Neumann series solution for hyperbolic automorphisms

Since a finitary solution of (1) for arbitrary base maps U has eluded us, it is interesting that for a rather general class of ergodic automorphisms (1) can be finitarily solved for every U. This method was pointed out to us by W. Parry.

Let G be a compact abelian group with translation invariant metric ρ. Say that an automorphism S of G is hyperbolic if there are almost continuous functions $\pi_s: G \to G$, $\pi_u: G \to G$ with ranges K_s and K_u, respectively, and constants $C > 0$ and $\lambda \in (0,1)$ such that

(i) $g = \pi_s(g) + \pi_u(g)$ for all $g \in G$,

(ii) $\rho(0,S^n g) < C\lambda^n$ if $g \in K_s$, $n \geq 0$,

(iii) $\rho(0,S^{-n} g) < C\lambda^n$ if $g \in K_u$, $n \geq 0$.

If S is a hyperbolic automorphism of $\mathbb{T}^n$ in the usual sense (no eigenvalues of modulus one), then it is hyperbolic in the above sense. For π_u can be obtained by embedding $\mathbb{T}^n$ into $\mathbb{R}^n$, projecting $\mathbb{R}^n$ along the stable eigenspace of S to the unstable eigenspace, and projecting back to $\mathbb{T}^n$. Similarly for π_s. Shifts on compact groups (e.g. the n-shift) are hyperbolic, where the maps π_s and π_u are to the future and to the past. Irreducible solenoidal automorphisms with no eigenvalues of modulus one are hyperbolic (see [3]). However, toral automorphisms with eigenvalues of modulus one and automorphisms of the full solenoid $\hat{\mathbb{R}}^n$ are not hyperbolic because they have isometric parts (as one can show).

We remark that hyperbolic group automorphisms are automatically ergodic. This follows because nonergodic automorphisms have a nontrivial isometric factor automorphism. By (ii) and (iii) K_s and K_u would map to the identity, violating (i).

<u>Theorem 3</u>: If S is a hyperbolic group automorphism of G and U is a map of X (not assumed compressible or even ergodic), then for each almost continuous function $\alpha: X \to G$ there is an almost continuous solution β to (1).

<u>Proof</u>: Let π_s, π_u, K_s, K_u, C, and λ be as in the definition of hyperbolicity for S.

We first claim that if we can solve (1) for α replaced by a translate $\alpha + g_0$, then we can solve it for α. For since S is ergodic, $(I-S)G = G$, so there is a $g_1 \in G$ such that $g_1 - Sg_1 = g_0$. If β is a solution for $\alpha + g_0$, then $\beta - g_1$ is a solution for α. Therefore an averaging argument as in the proof of the Corollary over translates of α shows that we may assume that $\alpha_s = \pi_s : X \to K_s$ and $\alpha_u = \pi_u\alpha : X \to K_u$ are almost continuous.

Now we just write down the solutions. Let

$$\beta_s(x) = \sum_{j=0}^{\infty} S^j \alpha_s(U^{-j-1}x),$$

$$\beta_u(x) = -\sum_{j=0}^{\infty} S^{-j-1}\alpha_u(U^j x).$$

Since $\alpha_s(U^{-j-1}x) \in K_s$, $\rho(0, S^j\alpha_s(U^{-j-1}x)) < C\lambda^j$, $j \geq 0$. Thus the series defining β_s converges uniformly where defined, and since $\alpha_s(U^{-j-1}x)$ is almost continuous, so is β_s. Similarly for β_u. An easy calculation shows that

$$\alpha_s(x) = \beta_s(Ux) - S\beta_x(x), \qquad \alpha_u(x) = \beta_u(Ux) - S\beta_s(x).$$

Since $\alpha(x) = \alpha_s(x) + \alpha_u(x)$, the function $\beta(x) = \beta_s(x) + \beta_u(x)$ is an almost continuous solution of (1).

The motivation for this solution and justification for the reference to Neumann series is as follows. Suppose S is a hyperbolic toral automorphism, and lift α to $\hat{\alpha}\colon X \to \mathbb{R}^n$. Form π_s projecting $\mathbb{R}^n$ to the stable eigenspace E^s as suggested above. On the Hilbert space $L^2(X, E^s)$ there are commuting operators $(\hat{U}f)(x) = f(Ux)$ and $(Sf)(x) = S(f(x))$. Then $\hat{U}$ is an isometry while $\|S\| < 1$. We are to solve

$$(\hat{U}-S)\beta_s = \alpha_s.$$

Thus

$$\begin{aligned}
\beta_x &= (\hat{U}-S)^{-1}\alpha_s \\
&= \hat{U}^{-1}(I-\hat{U}^{-1}S)^{-1}\alpha_s \\
&= \hat{U}^{-1}\sum_{j=0}^{\infty}(\hat{U}^{-1}S)^j\alpha_s \\
&= \sum_{j=0}^{\infty}\hat{U}^{-j-1}S^j\alpha_s.
\end{aligned}$$

Evaluating at x gives the definition of $\beta_s(x)$ above. A similar idea works for β_u by expanding

$$(\hat{U}-S)^{-1} = -S^{-1}(I-\hat{U}S^{-1})^{-1},$$

where $\|\hat{U}S^{-1}\|_{L^2(X,E^u)} < 1.$

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