Ergod. Th. & Dynam. Sys. (2001), **21**, 1695–1729
DOI: 10.1017/S014338570100181X

Expansive subdynamics for algebraic $\mathbb{Z}^d$-actions

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(*Received* 14 *June* 2000 *and accepted in revised form* 27 *September* 2000)

Abstract. A general framework for investigating topological actions of $\mathbb{Z}^d$ on compact metric spaces was proposed by Boyle and Lind in terms of expansive behavior along lower-dimensional subspaces of $\mathbb{R}^d$. Here we completely describe this expansive behavior for the class of algebraic $\mathbb{Z}^d$-actions given by commuting automorphisms of compact abelian groups. The description uses the logarithmic image of an algebraic variety together with a directional version of Noetherian modules over the ring of Laurent polynomials in several commuting variables.

We introduce two notions of rank for topological $\mathbb{Z}^d$-actions, and for algebraic $\mathbb{Z}^d$-actions describe how they are related to each other and to Krull dimension. For a linear subspace of $\mathbb{R}^d$ we define the group of points homoclinic to zero along the subspace, and prove that this group is constant within an expansive component.

1. *Introduction*

Expansiveness is a multifaceted property that plays an important role throughout dynamics. Let β be an action of $\mathbb{Z}^d$ by homeomorphisms of a compact metric space (X, ρ). Then β is called *expansive* if there is a $\delta > 0$ such that if x and y are distinct points of X then there is an $\mathbf{n} \in \mathbb{Z}^d$ such that $\rho(\beta^{\mathbf{n}}x, \beta^{\mathbf{n}}y) > \delta$.

In [**BL**] the notion of expansiveness along a subset, and especially a subspace, of $\mathbb{R}^d$ was introduced, by considering only those elements of $\mathbb{Z}^d$ that lie within a given bounded distance of the set. Let G_k denote the compact Grassmann manifold of k-dimensional subspaces (or *k-planes*) in $\mathbb{R}^d$, and $\mathsf{N}_k(\beta)$ be the set of those k-planes which are not expansive for β. It was shown in [**BL**] that if X is infinite, then $\mathsf{N}_{d-1}(\beta)$ is a non-empty compact subset of G_{d-1} that determines all other $\mathsf{N}_k(\beta)$ as follows: a k-plane is non-expansive for β if and only if it is contained in some subspace in $\mathsf{N}_{d-1}(\beta)$.

Denote by $\mathsf{E}_k(\beta)$ the set of expansive k-planes for β, which is an open subset of G_k. Various dynamical notions, such as entropy, can be defined along subspaces. The expansive subdynamics philosophy advocated in [**BL**] proposes that many such properties should be either constant or vary nicely within a connected component of $\mathsf{E}_k(\beta)$, but that they should typically change abruptly when passing from one component to another, analogous to a 'phase transition'. Several examples of this philosophy in action are given in [**BL**]. In §9 we provide another by considering points homoclinic along subspaces. Thus a basic starting point in the analysis of any topological $\mathbb{Z}^d$-action is to describe its expansive subspaces, especially those of co-dimension one, since these determine the rest.

An *algebraic* $\mathbb{Z}^d$*-action* is an action α of $\mathbb{Z}^d$ by (continuous) automorphisms of a compact abelian group, which we assume to be metrizable. We will consistently use α to denote an algebraic $\mathbb{Z}^d$-action and β for a general topological $\mathbb{Z}^d$-action. Such algebraic actions have provided a rich source of examples and phenomena (see [**S**]). The purpose of this paper is to completely determine the expansive subspaces for all algebraic $\mathbb{Z}^d$-actions.

In §2 we review the relevant ideas from [**BL**], and show that it is sufficient to use half-spaces rather than $(d-1)$-dimensional planes. The algebra we need is described in §3. In §4 we develop our main result, Theorem 4.9, which describes expansive half-spaces in terms of prime ideals. We give in §5 a number of examples that illustrate and motivate our results from the previous section. In §6 we investigate the prime ideal case in more detail, including an algorithm to compute the expansive set. We introduce two notions of 'rank' for a topological $\mathbb{Z}^d$-action in §7, and for algebraic $\mathbb{Z}^d$-actions show how they are related to each other and to Krull dimension. In §8 we extend our basic results to lower-dimensional subspaces of $\mathbb{R}^d$. The homoclinic group along a subspace is defined in §9 and shown to be constant within an expansive component. This fact has some interesting dynamical consequences.

2. *Expansive subdynamics*

Let (X, ρ) be a compact metric space, which we assume is infinite unless otherwise stated. A $\mathbb{Z}^d$*-action* β on X is a homomorphism from $\mathbb{Z}^d$ to the group of homeomorphisms of X. For $\mathbf{n} \in \mathbb{Z}^d$ we denote the corresponding homeomorphism by $\beta^{\mathbf{n}}$, so that $\beta^{\mathbf{m}} \circ \beta^{\mathbf{n}} = \beta^{\mathbf{m}+\mathbf{n}}$, and $\beta^{\mathbf{0}}$ is the identity on X. For a subset F of $\mathbb{R}^d$ put

$$\rho_\beta^F(x, y) = \sup\{\rho(\beta^{\mathbf{n}}(x), \beta^{\mathbf{n}}(y)) : \mathbf{n} \in F \cap \mathbb{Z}^d\},$$

and if $F \cap \mathbb{Z}^d = \varnothing$ define $\rho_\beta^F(x, y) = 0$.

Definition 2.1. A $\mathbb{Z}^d$-action β on (X, ρ) is *expansive* provided there is a $\delta > 0$ such that $\rho_\beta^{\mathbb{R}^d}(x, y) \le \delta$ implies that $x = y$. In this case δ is called an *expansive constant* for β.

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$. For $F \subset \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$ define $\mathrm{dist}(\mathbf{x}, F) = \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in F\}$. For $t > 0$ put $F^t = \{\mathbf{x} \in \mathbb{R}^d : \mathrm{dist}(\mathbf{x}, F) \le t\}$, so that F^t is the result of thickening F by t.

Definition 2.2. Let β be a $\mathbb{Z}^d$-action on (X, ρ) and F be a subset of $\mathbb{R}^d$. Then F is *expansive for* β, or β is *expansive along* F, if there are $\varepsilon > 0$ and $t > 0$ such that $\rho_\beta^{F^t}(x, y) \leq \varepsilon$ implies that $x = y$. If F fails to meet this condition it is *non-expansive for* β, or β is *non-expansive along* F.

Remark 2.3. Every subset of a non-expansive set for β is clearly also non-expansive for β. Every translate of an expansive set is expansive [**BL**, p. 57]. In the above definition we can take for ε a fixed expansive constant for β [**BL**, Lemma 2.3].

Next we examine subsets F that are linear subspaces of $\mathbb{R}^d$. Let $\mathsf{G}_k = \mathsf{G}_{d,k}$ denote the Grassmann manifold of k-dimensional subspaces (or simply *k-planes*) of $\mathbb{R}^d$. Recall that G_k is a compact manifold of dimension $k(d-k)$ whose topology is given by declaring two subspaces to be close if their intersections with the unit sphere are close in the Hausdorff metric. A k-plane and its $(d-k)$-dimensional orthogonal complement determine each other, giving a natural homeomorphism between G_k and G_{d-k}.

Definition 2.4. For a $\mathbb{Z}^d$-action β define

$$\mathsf{E}_k(\beta) = \{V \in \mathsf{G}_k : V \text{ is expansive for } \beta\},$$
$$\mathsf{N}_k(\beta) = \{V \in \mathsf{G}_k : V \text{ is non-expansive for } \beta\}.$$

An *expansive component of k-planes for* β is a connected component of $\mathsf{E}_k(\beta)$.

Example 2.5. (Ledrappier's example) Take $d = 2$,

$$X = \{x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j} + x_{i,j+1} \equiv 0 \pmod 2 \text{ for all } i, j\},$$

and let β be the $\mathbb{Z}^2$-action generated by the horizontal and vertical shifts. If L is a line that is not parallel to one of the sides of the unit simplex in $\mathbb{R}^2$ and $t \geq 2$, then for each $x \in X$ the coordinates of x within L^t determine all of x, so that $L \in \mathsf{E}_1(\beta)$. On the other hand, the three lines parallel to the sides of the simplex do not have this property, and they comprise $\mathsf{N}_1(\beta)$ (see [**BL**, Example 2.7] for details).

Simple coding arguments [**BL**, Lemma 3.4] show that each $\mathsf{E}_k(\beta)$ is an open subset of G_k, so that each $\mathsf{N}_k(\beta)$ is compact. Hence expansive components of k-planes for β are open subsets of G_k. By Remark 2.3, if W is non-expansive for β and V is a subspace of W, then V is also non-expansive for β. A basic result [**BL**, Theorem 3.6] is the converse: if V is a non-expansive subspace for β of dimension less than or equal to $d-2$, then there is a non-expansive subspace for β containing V of one higher dimension. If X is infinite, then the zero subspace is non-expansive, and hence inductively we see that each $\mathsf{N}_k(\beta) \neq \varnothing$ for $1 \leq k \leq d-1$. Furthermore, it follows that if $V \in \mathsf{N}_k(\beta)$, then there is a $W \in \mathsf{N}_{d-1}(\beta)$ that contains V. Hence $\mathsf{N}_k(\beta)$ consists of exactly all k-dimensional subspaces of the subspaces in $\mathsf{N}_{d-1}(\beta)$. Thus $\mathsf{N}_{d-1}(\beta)$ determines the entire expansive subdynamics of β.

In order to treat algebraic $\mathbb{Z}^d$-actions, it is convenient to shift our viewpoint slightly and use half-spaces in $\mathbb{R}^d$ rather than $(d-1)$-planes. Let $\mathsf{S}_{d-1} = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| = 1\}$ be the unit $(d-1)$-sphere. For $\mathbf{v} \in \mathsf{S}_{d-1}$ define $H_\mathbf{v} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{v} \leq 0\}$ to be the half-space with outward unit normal $\mathbf{v}$. Let H_d be the set of half-spaces in $\mathbb{R}^d$, which we identify with S_{d-1} via the parametrization $\mathbf{v} \leftrightarrow H_\mathbf{v}$. For $H \in \mathsf{H}_d$ we denote its outward unit normal vector by $\mathbf{v}_H$.

Expansiveness along a half-space H is defined using Definition 2.4 with $F = H$. Observe that thickening $H_{\mathbf{v}}$ by $t > 0$ results merely in the translation $H_{\mathbf{v}} + t\mathbf{v}$ of $H_{\mathbf{v}}$. Hence there is no need to thicken half-spaces in the definition, and a $\mathbb{Z}^d$-action β is therefore expansive along H if and only if there is an $\varepsilon > 0$ such that $\rho_\beta^H(x, y) \leq \varepsilon$ implies that $x = y$.

Definition 2.6. For a $\mathbb{Z}^d$-action β define

$$\begin{aligned} \mathsf{E}(\beta) &= \{H \in \mathsf{H}_d : H \text{ is expansive for } \beta\}, \\ \mathsf{N}(\beta) &= \{H \in \mathsf{H}_d : H \text{ is non-expansive for } \beta\}. \end{aligned}$$

An *expansive component of half-spaces for* β is a connected component of $\mathsf{E}(\beta)$.

Remark 2.7. A coding argument analogous to [**BL**, Lemma 3.4] shows that $\mathsf{E}(\beta)$ is an open set and so $\mathsf{N}(\beta)$ is a compact set.

The following lemma shows that a $(d-1)$-plane is non-expansive for β if and only if at least one of the two bounding half-spaces is also non-expansive for β. Thus if we define $\pi : \mathsf{H}_d \to \mathsf{G}_{d-1}$ by $\pi(H) = \partial H$, then $\pi(\mathsf{N}(\beta)) = \mathsf{N}_{d-1}(\beta)$. This shows that the half-space behavior $\mathsf{N}(\beta)$ determines the expansive subdynamics of β.

We start by recalling the following key notion from [**BL**, Definition 3.1].

Definition 2.8. Let β be an expansive $\mathbb{Z}^d$-action with expansive constant δ. For subsets E, F of $\mathbb{R}^d$ we say that E *codes* F provided that, for every $\mathbf{x} \in \mathbb{R}^d$, if $\rho_\beta^{E+\mathbf{x}}(x, y) \leq \delta$ then $\rho_\beta^{F+\mathbf{x}}(x, y) \leq \delta$.

LEMMA 2.9. *Let β be a $\mathbb{Z}^d$-action and $V \in \mathsf{G}_{d-1}$. Then $V \in \mathsf{N}_{d-1}(\beta)$ if and only if there is an $H \in \mathsf{N}(\beta)$ with $\partial H = V$.*

Proof. If $H \in \mathsf{N}(\beta)$, then by Remark 2.3 we see that $V = \partial H \subset H$ is also non-expansive.

Conversely, let $V \in \mathsf{G}_{d-1}$ and $H = H_{\mathbf{v}}$, $H' = H_{-\mathbf{v}}$ be the two half-spaces with boundary V. Suppose that both H and H' are expansive for β. We prove that V is also expansive for β, which will complete the proof.

Since β has an expansive half-space, it is an expansive action. Let $\delta > 0$ be an expansive constant for β. Let $B(r)$ denote the ball of radius r in $\mathbb{R}^d$, and $[\mathbf{0}, \mathbf{v}]$ be the line segment joining $\mathbf{0}$ to $\mathbf{v}$. A 'finite' version of the expansiveness of H, entirely analogous to [**BL**, Lemma 3.2], is that there is an $r > 0$ such that $H \cap B(r)$ codes $[\mathbf{0}, \mathbf{v}]$. Similarly, there is an $s > 0$ such that $H' \cap B(s)$ codes $[\mathbf{0}, -\mathbf{v}]$. Hence if $t = \max\{r, s\}$, then V^t codes V^{t+1}, which by the same argument codes V^{t+2}, and so on. Thus V^t codes $\mathbb{R}^d$, which means that V is expansive. □

3. *Algebraic $\mathbb{Z}^d$-actions*

An *algebraic $\mathbb{Z}^d$-action* is an action of $\mathbb{Z}^d$ by (continuous) automorphisms of a compact abelian group. Such actions provide a rich class of examples of $\mathbb{Z}^d$-actions having striking connections with commutative algebra. The monograph by Schmidt [**S**] provides a detailed account of this theory. For background and standard results from commutative algebra used below the reader may consult [**E**].

Let X be a compact abelian group with identity element 0_X. Suppose that α is an action of $\mathbb{Z}^d$ by automorphisms of X. Let $M = \widehat{X}$, the Pontryagin dual group of X. Define $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$, the ring of Laurent polynomials with integer coefficients in the d commuting variables $u_1, \ldots, u_d$. We can make M into an R_d-module by defining $u_j \cdot m = \widehat{\alpha}^{\mathbf{e}_j}(m)$ for all $m \in M$, where $\mathbf{e}_j \in \mathbb{Z}^d$ is the jth unit vector and $\widehat{\alpha}^{\mathbf{e}_j}$ is the automorphism of M dual to $\alpha^{\mathbf{e}_j}$. An element $f \in R_d$ has the form $f = f(u) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}}$, where the $c_f(\mathbf{n}) \in \mathbb{Z}$ and $c_f(\mathbf{n}) = 0$ for all but finitely many $\mathbf{n} \in \mathbb{Z}^d$, and $u^{\mathbf{n}} = u_1^{n_1} \ldots u_d^{n_d}$. Then $f \cdot m = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \widehat{\alpha}^{\mathbf{n}}(m)$ for every $m \in M$.

This process can be reversed. Suppose that M is an R_d-module. Let $X_M = \widehat{M}$ be its compact abelian dual group. Each u_j is a unit in R_d, so the map γ_j defined by $\gamma_j(m) = u_j \cdot m$ is an automorphism of M. Define an algebraic $\mathbb{Z}^d$-action α_M on X_M by $\alpha_M^{\mathbf{e}_j} = \widehat{\gamma}_j$. See [**S**, Ch. II] for further explanation and many examples.

Hence using duality we see there is a one-to-one correspondence between algebraic $\mathbb{Z}^d$-actions on the one hand and R_d-modules on the other.

A module over an arbitrary ring is *Noetherian* if it satisfies the ascending chain condition for submodules. The ring R_d is Noetherian as a module over itself. A prime ideal $\mathfrak{p} \subset R_d$ is *associated with* M if there is an $m \in M$ with $\mathfrak{p} = \{f \in R_d : f \cdot m = 0\}$. Let $\operatorname{asc}(M)$ denote the set of prime ideals associated with an R_d-module M. If M is Noetherian, then $\operatorname{asc}(M)$ is finite. One basic discovery has been that the dynamical properties of α_M can largely be determined from $\operatorname{asc}(M)$.

For example, let us describe when α_M is expansive. In order to do so, we need some notation. Let $\mathbb{C}$ denote the complex numbers and $\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\}$. For an ideal $\mathfrak{a}$ in R_d put

$$\mathsf{V}(\mathfrak{a}) = \{\mathbf{z} = (z_1, \ldots, z_d) \in (\mathbb{C}^\times)^d : f(z_1, \ldots, z_d) = 0 \text{ for all } f \in \mathfrak{a}\}$$

(we omit 0 from $\mathbb{C}$ since we are using Laurent polynomials). Let $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{S}^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\}$ be the multiplicative d-torus. Then we have the following characterization of expansiveness due to Schmidt (see [**S**, Theorem 6.5]), which can be thought of as a generalization of the fact that a toral automorphism is expansive if and only if the associated integer matrix has no eigenvalue in $\mathbb{S}$.

THEOREM 3.1. *Let M be an R_d-module. Then the following are equivalent:*
(1) *α_M is expansive;*
(2) *M is Noetherian and $\alpha_{R_d/\mathfrak{p}}$ is expansive for every $\mathfrak{p} \in \operatorname{asc}(M)$;*
(3) *M is Noetherian and $\mathsf{V}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$ for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

Since the expansive subdynamics of a non-expansive action are trivial, we will consider only Noetherian R_d-modules from now on. In particular, a Noetherian R_d-module M is countable, and hence X_M is metrizable. We fix a metric ρ on X_M, which we may assume to be translation-invariant. Observe that $\rho(\alpha^{\mathbf{n}}(x), \alpha^{\mathbf{n}}(y)) = \rho(\alpha^{\mathbf{n}}(x-y), \alpha^{\mathbf{n}}(0_X))$ by our assumption on ρ. Thus in considering the expansive behavior of an algebraic $\mathbb{Z}^d$-action on a pair of points, we may assume that one of them is 0_X.

Roughly speaking, non-expansiveness of α_M can occur for two reasons: algebraic and geometric. The algebraic reason occurs when M is not Noetherian: in this case there is a decreasing sequence of closed, α_M-invariant subgroups converging to $\{0_X\}$, and this immediately provides, for every $\delta > 0$, a non-zero point that remains within δ of 0_X under all iterates of α_M. The geometric reason occurs when $\mathsf{V}(\mathfrak{p}) \cap \mathbb{S}^d \neq \varnothing$: in this case it is possible to use a element from $\mathsf{V}(\mathfrak{p}) \cap \mathbb{S}^d$ to construct points that remain arbitrarily close to 0_X under all iterates (see the proof of Theorem 4.9 for details). For a valuation-theoretic approach to expansive behavior see [**M**].

4. *Characterization of expansive half-spaces*

Let M be an R_d-module and α_M be the corresponding algebraic $\mathbb{Z}^d$-action. In this section we characterize those half-spaces $H \in \mathsf{H}_d$ which are expansive for α_M in terms of the prime ideals associated with M. The main result, Theorem 4.9, is a 'one-sided' version of Theorem 3.1. The reader is urged to consult the examples in §5 first to motivate what follows in this section.

According to Theorem 3.1, there are two reasons that α_M may fail to be expansive: M may not be Noetherian, or there may be a point in $\mathsf{V}(\mathfrak{p}) \cap \mathbb{S}^d$ for some $\mathfrak{p} \in \operatorname{asc}(M)$. In the following sequence of results we investigate a 'one-sided' version of each of these possibilities. Our proofs closely parallel those in [**S**], with suitable modifications for their one-sided nature.

We start with the Noetherian condition. For $H \in \mathsf{H}_d$, recall that $\mathbf{v}_H$ denotes the outward unit normal for H. Define $H_{\mathbb{Z}} = H \cap \mathbb{Z}^d$. Put $R_H = \mathbb{Z}[u^{\mathbf{n}} : \mathbf{n} \in H_{\mathbb{Z}}]$, which is a subring of R_d. It is important to note that in general R_H is *not* a Noetherian ring. Indeed, R_H is Noetherian exactly when $\mathbf{v}_H$ is a rational direction in the sense that $\mathbb{R}\mathbf{v}_H \cap \mathbb{Z}^d \neq \{\mathbf{0}\}$, so that R_H is Noetherian for only countably many H. Understanding when an R_d-module is Noetherian as an R_H-module (i.e. when it is R_H-*Noetherian*) is one of the key points in our analysis.

LEMMA 4.1. *Let M be a Noetherian R_d-module and $H \in \mathsf{H}_d$. Then M is R_H-Noetherian if and only if $R_d/\mathfrak{p}$ is R_H-Noetherian for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

Proof. Suppose that M is R_H-Noetherian. If $\mathfrak{p} \in \operatorname{asc}(M)$, then there is an $m \in M$ such that $R_d \cdot m \cong R_d/\mathfrak{p}$. By definition, every R_H-submodule of M is R_H-Noetherian, and in particular so is $R_d \cdot m$. Hence $R_d/\mathfrak{p}$ is R_H-Noetherian for every $\mathfrak{p} \in \operatorname{asc}(M)$.

Conversely, suppose that $R_d/\mathfrak{p}$ is R_H-Noetherian for every $\mathfrak{p} \in \operatorname{asc}(M)$. Since M is R_d-Noetherian, there is a chain of R_d-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$$

with $M_j/M_{j-1} \cong R_d/\mathfrak{q}_j$ for $1 \le j \le r$, where each $\mathfrak{q}_j$ is a prime ideal in R_d that contains some $\mathfrak{p}_j \in \operatorname{asc}(M)$ [**S**, Corollary 6.2]. The surjection $R_d/\mathfrak{p}_j \to R_d/\mathfrak{q}_j$ shows that $R_d/\mathfrak{q}_j$ is again R_H-Noetherian. Recall the fact from module theory that over an arbitrary ring if $P \subset Q$ are modules, then Q is Noetherian if and only if both P and Q/P are Noetherian. Repeated application of this fact shows successively that $M_1, M_2, \ldots,$ and finally $M_r = M$ are R_H-Noetherian. □

We will use twice the following variant of the 'determinant trick,' which for clarity we state separately.

LEMMA 4.2. *Let $\mathcal{R}$ be a commutative ring with unit and $\mathcal{M}$ be an $\mathcal{R}$-module generated by $b_1, \dots, b_k$. Suppose that $r_{ij} \in \mathcal{R}$ for $1 \le i, j \le k$ are such that $\sum_{j=1}^{k} r_{ij} b_j = 0$ for $1 \le i \le k$. Then* $\det[r_{ij}]$ *annihilates $\mathcal{M}$.*

Proof. Consider the matrix $A = [r_{ij}]$ acting on column vectors in $\mathcal{M}^k$. If $\mathbf{b} \in \mathcal{M}^k$ has jth entry b_j, then $A\mathbf{b} = 0$. Multiplying this equation by the adjugate matrix of A gives $(\det A)\mathbf{b} = 0$. Hence $(\det A)b_j = 0$ for $1 \le j \le k$, so that $\det A$ annihilates $\mathcal{M}$. □

In order to show that expansiveness of α_M along H implies that M is R_H-Noetherian, we need one further algebraic result. Recall that modules over Noetherian rings are Noetherian exactly when they are finitely generated. This fails for non-Noetherian rings. For example, if R_H is non-Noetherian, then R_H is finitely generated over itself (by 1), yet the ideal $\mathbb{Z}[u^{\mathbf{n}} : \mathbf{n}\cdot\mathbf{v}_H < 0]$ is not finitely generated over R_H. Nevertheless, in our situation there is a reasonable substitute.

LEMMA 4.3. *Let M be an R_d-module and $H \in \mathsf{H}_d$. Then M is R_H-Noetherian if and only if M is finitely generated over R_H.*

Proof. Noetherian modules over arbitrary rings are always finitely generated.

Conversely, suppose that M is finitely generated over R_H, say by $m_1, \dots, m_r$. Fix $\mathbf{k} \in \mathbb{Z}^d \smallsetminus H$. Since M is an R_d-module, we can find $f_{ij}(u) \in R_H$ such that

$$u^{\mathbf{k}} \cdot m_j = \sum_{i=1}^{r} f_{ij}(u) \cdot m_i.$$

If I denotes the $r \times r$ identity matrix and $F = [f_{ij}(u)]$, then Lemma 4.2 shows that $\det(u^{\mathbf{k}} I - F)$ annihilates M. Multiplying this determinant by $u^{-(r-1)\mathbf{k}}$ shows that there is an element in R_d of the form $u^{\mathbf{k}} - f(u)$, with $f(u) \in R_H$, that annihilates M.

Suppose that N is an R_H-submodule of M. For every $n \in N \subset M$ we have that $(u^{\mathbf{k}} - f(u)) \cdot n = 0$, so that $u^{\mathbf{k}} \cdot n = f(u) \cdot n \in N$. Hence N is closed under the subring of R_d generated by R_H and $u^{\mathbf{k}}$, which is all of R_d. Thus every R_H-submodule of M is also an R_d-submodule. Since M is finitely generated over R_H, it is finitely generated over R_d, hence M is R_d-Noetherian. It then follows that M is also R_H-Noetherian (since every R_H-submodule is also an R_d-submodule). □

The proof of Lemma 4.3 also establishes the following useful result.

LEMMA 4.4. *Let M be a Noetherian R_d-module, $H \in \mathsf{H}_d$, and $\mathbf{k} \in \mathbb{Z}^d \smallsetminus H$. Then M is R_H-Noetherian if and only if there is a polynomial of the form $u^{\mathbf{k}} - f(u)$ with $f(u) \in R_H$ that annihilates M.*

Remark 4.5. If M is a Noetherian R_d-module, Lemma 4.4 shows that $\{H \in \mathsf{H}_d : M \text{ is } R_H\text{-Noetherian}\}$ is an open subset of H_d. To see this, suppose that M is R_H-Noetherian for some $H \in \mathsf{H}_d$. Fix $\mathbf{k} \in \mathbb{Z}^d \smallsetminus H$. Applying Lemma 4.4 with $\mathbf{k}$ replaced by $2\mathbf{k}$, we see there is an $f(u) \in R_H$ such that $u^{2\mathbf{k}} - f(u)$ annihilates M. Then $u^{\mathbf{k}} - u^{-\mathbf{k}} f(u)$ also annihilates M. For H' sufficiently close to H it follows that $u^{-\mathbf{k}} f(u)$ is also in $R_{H'}$ and that $\mathbf{k} \in \mathbb{Z}^d \smallsetminus H'$, and hence by another application of Lemma 4.4 we find that M is $R_{H'}$-Noetherian as well.

LEMMA 4.6. *Let M be an R_d-module and $H \in \mathsf{H}_d$. If α_M is expansive along H then M is R_H-Noetherian.*

Proof. If M is not R_H-Noetherian, then by Lemma 4.3 it is not finitely generated over R_H. Hence there is a strictly increasing sequence of proper R_H-submodules $M_1 \subsetneqq M_2 \subsetneqq \cdots$ with $\bigcup_{j=1}^{\infty} M_j = M$. Let $X_j = M_j^{\perp} \subset X_M$. Then each X_j is compact, $X_1 \supsetneqq X_2 \supsetneqq \cdots$, and $\bigcap_{j=1}^{\infty} X_j = \{0_X\}$. Hence $\operatorname{diam}(X_j) \to 0$. Furthermore, since M_j is an R_H-module, $\alpha_M^{\mathbf{n}}(X_j) \subset X_j$ for every $\mathbf{n} \in H_{\mathbb{Z}}$.

Let $\varepsilon > 0$. Choose j_0 such that $\operatorname{diam}(X_{j_0}) < \varepsilon$, and pick $0 \neq x \in X_{j_0}$. Then $\alpha_M^{\mathbf{n}}(x) \in X_{j_0}$ for all $\mathbf{n} \in H_{\mathbb{Z}}$, so that $\sup\{\rho(\alpha_M^{\mathbf{n}}(x), 0_X) : \mathbf{n} \in H_{\mathbb{Z}}\} < \varepsilon$. Since ε was arbitrary, we see that α_M is not expansive along H. This contradiction proves that M is R_H-Noetherian. □

Next we turn to the one-sided version of the variety condition. Define $\log|\mathbf{z}|$ for $\mathbf{z} = (z_1, \ldots, z_d) \in (\mathbb{C}^{\times})^d$ by

$$\log|\mathbf{z}| = (\log|z_1|, \ldots, \log|z_d|) \in \mathbb{R}^d.$$

If $\mathfrak{a}$ is an ideal in R_d, put

$$\log|\mathsf{V}(\mathfrak{a})| = \{\log|\mathbf{z}| : \mathbf{z} \in \mathsf{V}(\mathfrak{a})\} \subset \mathbb{R}^d.$$

When $\mathfrak{a} = \langle f \rangle$ is principal, the set $\log|V(\mathfrak{a})| = \log|\mathsf{V}(f)|$ was investigated in [**GKZ**, §6.1], where it is called the *amoeba* of f (turn to Figure 2 in §5 to see why). For example, they show that the connected components of the complement of $\log|\mathsf{V}(f)|$ are all convex sets that are in one-to-one correspondence with the distinct domains of convergence of Laurent expansions of $1/f$.

For $\mathbf{v} \in \mathsf{S}_{d-1}$, let $[0, \infty)\mathbf{v} = \{t\mathbf{v} : t \geq 0\}$ denote the ray in $\mathbb{R}^d$ through $\mathbf{v}$.

PROPOSITION 4.7. *Let $\mathfrak{a}$ be an ideal in R_d and $H \in \mathsf{H}_d$ with outward unit normal $\mathbf{v}_H$. Then $\alpha_{R_d/\mathfrak{a}}$ is expansive along H if and only if $R_d/\mathfrak{a}$ is R_H-Noetherian and $[0, \infty)\mathbf{v}_H \cap \log|\mathsf{V}(\mathfrak{a})| = \varnothing$.*

Proof. Let σ denote the $\mathbb{Z}^d$-shift action on $\mathbb{T}^{\mathbb{Z}^d}$, where $(\sigma^{\mathbf{n}}x)_{\mathbf{m}} = x_{\mathbf{n}+\mathbf{m}}$. For $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})u^{\mathbf{n}} \in R_d$ put $f(\sigma) = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})\sigma^{\mathbf{n}}$. Then $X_{R_d/\mathfrak{a}}$ is the closed shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$ given by

$$X_{R_d/\mathfrak{a}} = \{x \in \mathbb{T}^{\mathbb{Z}^d} : f(\sigma)x = 0 \text{ for all } f \in \mathfrak{a}\}.$$

If $f + \mathfrak{a} \in R_d/\mathfrak{a}$ and $x \in X_{R_d/\mathfrak{a}}$, the duality pairing is given by the formula

$$\langle f, x\rangle = \exp[2\pi i(f(\sigma)x)_{\mathbf{0}}].$$

First suppose that $\alpha_{R_d/\mathfrak{a}}$ is expansive along H. Lemma 4.6 shows that $R_d/\mathfrak{a}$ is R_H-Noetherian. If $[0, \infty)\mathbf{v}_H \cap \log|\mathsf{V}(\mathfrak{a})| \neq \varnothing$, choose $\mathbf{z} \in \mathsf{V}(\mathfrak{a})$ such that $(\log|\mathbf{z}|) \cdot \mathbf{n} \leq 0$ for all $\mathbf{n} \in H_{\mathbb{Z}}$. Consider $\mathbb{C}^{\mathbb{Z}^d}$ together with the $\mathbb{Z}^d$-shift action σ, and the point $w \in \mathbb{C}^{\mathbb{Z}^d}$ defined by $w_{\mathbf{n}} = \mathbf{z}^{\mathbf{n}} = z_1^{n_1} \cdots z_d^{n_d}$. For every $f \in R_d$ we have $f(\sigma)(w) = f(\mathbf{z})w$, and so $f(\sigma)(w) = 0$ for all $f \in \mathfrak{a}$. Fix $\varepsilon > 0$. Then $|\varepsilon w_{\mathbf{n}}| = |\varepsilon \mathbf{z}^{\mathbf{n}}| \leq \varepsilon$ for every $\mathbf{n} \in H_{\mathbb{Z}}$. Define $x \in \mathbb{T}^{\mathbb{Z}^d}$ by $x_{\mathbf{n}} = \operatorname{Re}(\varepsilon w_{\mathbf{n}}) \pmod 1$. Then clearly $x \neq 0$, and $|x_{\mathbf{n}}| \leq \varepsilon$ for all

$\mathbf{n} \in H_{\mathbb{Z}}$. Since ε was arbitrary, this contradicts expansiveness of $\alpha_{R_d/\mathfrak{a}}$ along H. Hence $[0, \infty)\mathbf{v}_H \cap \log |\mathsf{V}(\mathfrak{a})| = \varnothing$.

Conversely, suppose that $R_d/\mathfrak{a}$ is R_H-Noetherian. Lemma 4.4 shows that there is a polynomial g in $\mathfrak{a}$ of the form $g(u) = 1 - \sum_{\mathbf{n}\in G} c_g(\mathbf{n})u^{\mathbf{n}}$, where $\mathbf{n} \cdot \mathbf{v}_H < 0$ for all $\mathbf{n} \in G$. Now $\mathfrak{a}$ is finitely generated over R_d, say by $f_1, \dots, f_r$.

For $h = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_h(\mathbf{n})u^{\mathbf{n}} \in R_d$ put $\|h\| = \sum_{\mathbf{n}\in\mathbb{Z}^d} |c_h(\mathbf{n})|$. Define $\varepsilon = (10\|g\| + 10\sum_{j=1}^{r} \|f_j\|)^{-1}$. For $t \in \mathbb{T}$ let $|t| = \min\{|t - n| : n \in \mathbb{Z}\}$. We will prove that if $x \in X_{R_d/\mathfrak{a}}$ and $|x_{\mathbf{n}}| < \varepsilon$ for all $\mathbf{n} \in H_{\mathbb{Z}}$, then $x = 0$, which will show that $\alpha_{R_d/\mathfrak{a}}$ is expansive.

There are two cases to consider: (1) $x_{\mathbf{n}} = 0$ for all $\mathbf{n} \in H_{\mathbb{Z}}$, and (2) $x_{\mathbf{n}} \neq 0$ for some $\mathbf{n} \in H_{\mathbb{Z}}$.

In case (1), choose $\theta > 0$ so that $\mathbf{n} \cdot \mathbf{v}_H \leq -\theta$ for all $\mathbf{n} \in G$. If $\mathbf{k} \in \mathbb{Z}^d$ with $0 < \mathbf{k}\cdot\mathbf{v}_H \leq \theta$ then $(\mathbf{n}+\mathbf{k})\cdot\mathbf{v}_H \leq 0$ for $\mathbf{n} \in G$. Now $u^{\mathbf{k}}g(u) = u^{\mathbf{k}} - \sum_{\mathbf{n}\in G} c_g(\mathbf{n})u^{\mathbf{n}+\mathbf{k}} \in \mathfrak{a}$, so that $x_{\mathbf{k}} = \sum_{\mathbf{n}\in G} c_g(\mathbf{n})x_{\mathbf{n}+\mathbf{k}} = 0$ since $x_{\mathbf{n}+\mathbf{k}} = 0$ for all $\mathbf{n} \in G$ by assumption. Hence $x_{\mathbf{k}} = 0$ for all $\mathbf{k}$ with $0 < \mathbf{k} \cdot \mathbf{v}_H \leq \theta$. Repeating this argument shows that $x_{\mathbf{k}} = 0$ if $\mathbf{k} \cdot \mathbf{v}_H \leq 2\theta$, then if $\mathbf{k} \cdot \mathbf{v}_H \leq 3\theta$, and so on, which proves that $x = 0$.

We now turn to case (2). We will show that there is a point in $[0, \infty)\mathbf{v}_H \cap \log |\mathsf{V}(\mathfrak{a})|$. Consider the Banach space $\ell^\infty(H_{\mathbb{Z}})$ of all bounded complex-valued functions on $H_{\mathbb{Z}}$. For every $\mathbf{n} \in H_{\mathbb{Z}}$ the operator $U_{\mathbf{n}}$ defined by $(U_{\mathbf{n}}w)_{\mathbf{m}} = w_{\mathbf{m}+\mathbf{n}}$ maps $\ell^\infty(H_{\mathbb{Z}})$ to itself, and clearly $\|U_{\mathbf{n}}\| \leq 1$. For $f = \sum_{\mathbf{n}\in H_{\mathbb{Z}}} c_f(\mathbf{n})u^{\mathbf{n}} \in R_H$ define the operator $f(U) = \sum_{\mathbf{n}\in H_{\mathbb{Z}}} c_f(\mathbf{n})U_{\mathbf{n}}$. Put

$$W = \{w \in \ell^\infty(H_{\mathbb{Z}}) : U_{\mathbf{m}} f_j(U)w = 0 \text{ and } U_{\mathbf{m}} g(U)w = 0 \\ \text{for } 1 \leq j \leq r \text{ and all } \mathbf{m} \in H_{\mathbb{Z}}\}.$$

Clearly W is closed and mapped to itself by $U_{\mathbf{n}}$ for every $\mathbf{n} \in H_{\mathbb{Z}}$. We claim that W is non-trivial. For define $w \in W$ by taking $w_{\mathbf{n}}$ to be the unique number in $(-\varepsilon, \varepsilon)$ for which $x_{\mathbf{n}} \equiv w_{\mathbf{n}} \pmod 1$. Since

$$1 = \langle u^{\mathbf{m}} f_j(u), x\rangle = \exp[2\pi i(\sigma^{\mathbf{m}} f_j(\sigma)x)_{\mathbf{0}}],$$

if follows that $(\sigma^{\mathbf{m}} f_j(\sigma)w)_{\mathbf{0}} = \sum c_{f_j}(\mathbf{n})w_{\mathbf{m}+\mathbf{n}} \in \mathbb{Z}$ for all $\mathbf{m} \in H_{\mathbb{Z}}$. Our size condition on x coupled with $u^{\mathbf{m}} f_j(u) \in R_H$ shows that $\sum c_{f_j}(\mathbf{n})w_{\mathbf{m}+\mathbf{n}} = 0$ for $1 \leq j \leq r$ and $\mathbf{m} \in H_{\mathbb{Z}}$. The same argument works for g, proving that w is a non-zero element of W.

For each $\mathbf{n} \in H_{\mathbb{Z}}$ let $V_{\mathbf{n}}$ be the restriction of $U_{\mathbf{n}}$ to W. Let $\mathcal{A}$ be the commutative Banach algebra of bounded operators on W generated by $\{V_{\mathbf{n}} : \mathbf{n} \in H_{\mathbb{Z}}\}$. The theory of commutative Banach algebras shows that there is a (non-zero) complex homomorphism $\omega\colon \mathcal{A} \to \mathbb{C}$, and that $|\omega(V)| \leq \|V\|$ for all $V \in \mathcal{A}$. Let $a_{\mathbf{n}} = \omega(V_{\mathbf{n}})$ for all $\mathbf{n} \in H_{\mathbb{Z}}$. Then $\mathbf{n} \mapsto a_{\mathbf{n}}$ is a homomorphism from the monoid $H_{\mathbb{Z}}$ to the multiplicative monoid $\mathbb{C}$. It follows that there is $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d$ such that $a_{\mathbf{n}} = \mathbf{z}^{\mathbf{n}}$ for all $\mathbf{n} \in H_{\mathbb{Z}}$. Also, $f_j(V) = 0$ on W, so applying ω shows that $f_j(\mathbf{z}) = g(\mathbf{z}) = 0$.

We claim that $z_j \neq 0$ for $1 \leq j \leq d$. Since $\omega \neq 0$, there is a $\mathbf{k} \in H_{\mathbb{Z}}$ such that $a_{\mathbf{k}} \neq 0$. Suppose that $a_{\mathbf{n}} = 0$ for all $\mathbf{n} \in H_{\mathbb{Z}}$ with $\mathbf{n} \cdot \mathbf{v}_H < 0$. If $\mathbf{m} \in \partial H \cap \mathbb{Z}^d$, then $a_{\mathbf{m}} = \sum_{\mathbf{n}\in G} c_g(\mathbf{n})a_{\mathbf{m}+\mathbf{n}} = 0$, so that a is also zero on $H_{\mathbb{Z}}$, a contradiction. Hence there is an $\mathbf{n} \in H_{\mathbb{Z}}$ with $\mathbf{n} \cdot \mathbf{v}_H < 0$ and $a_{\mathbf{n}} \neq 0$. It then follows from multiplicativity of the $a_{\mathbf{n}}$ that

$a_{\mathbf{n}} \neq 0$ for all $\mathbf{n} \in H_{\mathbb{Z}}$. Hence $z_j \neq 0$ for all j, and so $\mathbf{z} \in \mathsf{V}(\mathfrak{a})$. Finally, for all $\mathbf{n} \in H_{\mathbb{Z}}$ we have that

$$|\mathbf{z}^{\mathbf{n}}| = |a_{\mathbf{n}}| = |\omega(V_{\mathbf{n}})| \leq \|V_{\mathbf{n}}\| \leq 1,$$

so that $\mathbf{n} \cdot \log|\mathbf{z}| \leq 0$, proving that $\log|\mathbf{z}| \in [0, \infty)\mathbf{v}_H$. □

We will need the following fact when dealing with general R_d-modules.

LEMMA 4.8. *Let α be an algebraic $\mathbb{Z}^d$-action on X, Y be an α-invariant compact subgroup of X, and $H \in \mathsf{H}_d$. If α_Y and $\alpha_{X/Y}$ are both expansive along H, then so is α.*

Proof. The hypothesis shows that there is a neighborhood U of 0_X in X such that $\bigcap_{\mathbf{n}\in H_{\mathbb{Z}}} \alpha^{\mathbf{n}}_{X/Y}(U+Y) = \{0_{X/Y}\}$ and $\bigcap_{\mathbf{n}\in H_{\mathbb{Z}}} \alpha^{\mathbf{n}}_{Y}(U \cap Y) = \{0_X\}$. These imply that $\bigcap_{\mathbf{n}\in H_{\mathbb{Z}}} \alpha^{\mathbf{n}}(U) = \{0_X\}$, so that α is expansive. □

We are now ready to characterize the expansive half-spaces for algebraic $\mathbb{Z}^d$-actions. First observe that by Theorem 3.1 if an R_d-module M is not R_d-Noetherian, then α_M is not expansive, and so all subsets of $\mathbb{R}^d$ are also non-expansive and we are done. Thus we can assume the modules considered are Noetherian over R_d.

THEOREM 4.9. *Let M be a Noetherian R_d-module, α_M be the corresponding algebraic $\mathbb{Z}^d$-action, and $H \in \mathsf{H}_d$. Then the following are equivalent:*

(1) *α_M is expansive along H;*

(2) *$\alpha_{R_d/\mathfrak{p}}$ is expansive along H for every $\mathfrak{p} \in \operatorname{asc}(M)$;*

(3) *$R_d/\mathfrak{p}$ is R_H-Noetherian and $[0, \infty)\mathbf{v}_H \cap \log|\mathsf{V}(\mathfrak{p})| = \varnothing$ for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

Proof. Proposition 4.7 shows that (2) ⇔ (3).

(3) ⇒ (1): There is a chain of R_d-submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that $M_j/M_{j-1} \cong R_d/\mathfrak{q}_j$, where $\mathfrak{q}_j$ is a prime ideal in R_d containing some $\mathfrak{p}_j \in \operatorname{asc}(M)$. Since $X_{R_d/\mathfrak{q}_j} \subset X_{R_d/\mathfrak{p}_j}$, we see that $\alpha_{R_d/\mathfrak{q}_j}$ is expansive along H. Put $X_j = M_j^{\perp} \subset X_M$. Then $X_M = X_0 \supset X_1 \supset \cdots X_r = 0$, and $X_{j-1}/X_j \cong X_{R_d/\mathfrak{q}_j}$. Repeated application of Lemma 4.8 shows successively that $\alpha_{X_{r-1}}, \alpha_{X_{r-2}}, \dots, \alpha_{X_0} = \alpha_M$ are all expansive along H.

(1) ⇒ (3): Suppose that α_M is expansive along H. By Lemma 4.6, M is R_H-Noetherian, so that by Lemma 4.1 we have that $R_d/\mathfrak{p}$ is R_H-Noetherian for every $\mathfrak{p} \in \operatorname{asc}(M)$.

Suppose that there is a $\mathfrak{p} \in \operatorname{asc}(M)$ for which $[0, \infty)\mathbf{v}_H \cap \log|\mathsf{V}(\mathfrak{p})| \neq \varnothing$. Fix $\mathbf{z} \in \mathsf{V}(\mathfrak{p})$ with $\log|\mathbf{z}| = t\mathbf{v}_H$ for some $t \geq 0$. Consider $\mathbb{C}$ as an R_d-module using the action θ defined by $\theta^{\mathbf{n}}(\xi) = \mathbf{z}^{\mathbf{n}}\xi$ for all $\xi \in \mathbb{C}$. We will construct an R_d-homomorphism $\psi\colon M \to \mathbb{C}$.

To construct ψ, first note that M is R_d-Noetherian by assumption, hence finitely generated over R_d. Choose generators $m_1, m_2, \dots, m_k$, and define the surjective map $\zeta\colon R_d^k \to M$ by $\zeta(f_1, \dots, f_k) = f_1 \cdot m_1 + \cdots + f_k \cdot m_k$. Let $K = \ker\zeta$. Define $\phi\colon R_d^k \to \mathbb{C}^k$ by $\phi(f_1, \dots, f_k) = (f_1(\mathbf{z}), \dots, f_k(\mathbf{z}))$. We claim that the dimension of the complex vector space generated by $\phi(K)$ is strictly less than k. For if not, there are elements $\mathbf{f}^{(1)} = (f_1^{(1)}, \dots, f_k^{(1)}), \dots, \mathbf{f}^{(k)} = (f_1^{(k)}, \dots, f_k^{(k)})$ in K such that $\phi(\mathbf{f}^{(1)}), \dots, \phi(\mathbf{f}^{(k)})$ are linearly independent in $\mathbb{C}^k$. Denote the $k \times k$ matrix $[f_j^{(i)}]$ by F. Since each $\mathbf{f}^{(i)} \in K$, Lemma 4.2 shows that $\det F$ annihilates M, and in particular $\det F \in \mathfrak{p}$. Thus

$$0 = (\det F)(\mathbf{z}) = \det[f_j^{(i)}(\mathbf{z})] = \det[\phi(\mathbf{f}^{(1)}), \dots, \phi(\mathbf{f}^{(k)})],$$

contradicting linear independence of the $\phi(\mathbf{f}^{(i)})$. This proves that $\phi(K)$ generates a proper complex vector subspace L of $\mathbb{C}^k$. Since $\phi(R_d^k)$ generates all of $\mathbb{C}^k$, it follows that the corresponding quotient map $\widetilde{\phi}\colon M \cong R_d^k/K \to \mathbb{C}^k/L$ is non-zero. We can therefore compose $\widetilde{\phi}$ with a projection of $\mathbb{C}^k/L$ to a 1-dimensional subspace so that the composition is still a non-zero R_d-homomorphism. The result is the desired $\psi : M \to \mathbb{C}$.

By construction, $\psi(f_j \cdot m_j) = c_j f_j(\mathbf{z})$. After multiplying (if necessary) by a constant we can assume that $|c_j| < \varepsilon$ and that the point $x \in X_M = \widehat{M}$ defined by $x(u^{\mathbf{n}} \cdot m_j) = \exp[2\pi i \operatorname{Re}(c_j \mathbf{z}^{\mathbf{n}})]$ is not trivial. Since $\alpha^{\mathbf{n}}(x)$ is close to 0_X for all $\mathbf{n} \in H_{\mathbb{Z}}$, the half-space H is not expansive. □

Remark 4.10. Roughly speaking, we can recover the statement (and proof) of Theorem 3.1 from the preceding theorem by using $\mathbf{v} = \mathbf{v}_H = 0$. For then $[0, \infty)\mathbf{v}_H = \{\mathbf{0}\}$, so that $[0, \infty)\mathbf{v}_H \cap \log|\mathsf{V}(\mathfrak{p})| = \varnothing$ if and only if $\mathsf{V}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$, and $H_{\mathbf{v}} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{v} \le 0\} = \mathbb{R}^d$.

The content of Theorem 4.9 can be summarized by the equalities

$$\mathsf{E}(\alpha_M) = \bigcap_{\mathfrak{p} \in \operatorname{asc}(M)} \mathsf{E}(\alpha_{R_d/\mathfrak{p}}), \quad \mathsf{N}(\alpha_M) = \bigcup_{\mathfrak{p} \in \operatorname{asc}(M)} \mathsf{N}(\alpha_{R_d/\mathfrak{p}}).$$

Our arguments would be substantially simplified if we knew that the converse of Lemma 4.8 were true, namely that the quotient of an algebraic $\mathbb{Z}^d$-action that is expansive along H is also expansive along H. Although this is correct, the only proof we know makes full use of Theorem 4.9.

PROPOSITION 4.11. *Let α be an algebraic $\mathbb{Z}^d$-action on X, Y be a compact α-invariant subgroup of X, and $H \in \mathsf{H}_d$. Then α is expansive along H if and only if both α_Y and $\alpha_{X/Y}$ are expansive along H.*

Proof. Lemma 4.8 proves one direction. For the other, suppose α is expansive along H. Then trivially α_Y is expansive along H. Let $N = Y^{\perp} \subset M$, so that $X/Y = X_N$. Now $\operatorname{asc}(N) \subset \operatorname{asc}(M)$, so Theorem 4.9 shows that $\alpha_N = \alpha_{X/Y}$ is expansive along H. □

5. *Examples*

This section contains a number of examples to illustrate the above ideas. One noteworthy feature is the elegant way in which the non-Noetherian and variety pieces of the non-expansive set fit together.

Our examples involve three or fewer variables, so for notational simplicity we use u, v, w, instead of u_1, u_2, u_3. Using the correspondence $\mathsf{H}_d \leftrightarrow \mathsf{S}_{d-1}$ given by $H \leftrightarrow \mathbf{v}_H$, we identify subsets of H_d with the corresponding subsets of S_{d-1} for ease of visualization. Using this convention, for an ideal $\mathfrak{a} \in R_d$ we put

$$\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{a}}) = \{\mathbf{v} \in \mathsf{S}_{d-1} : R_d/\mathfrak{a} \text{ is not } R_{H_{\mathbf{v}}}\text{-Noetherian}\},$$

$$\mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{a}}) = \{\mathbf{v} \in \mathsf{S}_{d-1} : [0, \infty)\mathbf{v} \cap \log|\mathsf{V}(\mathfrak{a})| \ne \varnothing\}.$$

Observe that $\mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{a}})$ is the radial projection of $\log|\mathsf{V}(\mathfrak{a})|$ to S_{d-1}. By Proposition 4.7,

$$\mathsf{N}(\alpha_{R_d/\mathfrak{a}}) = \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{a}}) \cup \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{a}}).$$

As before, in the case of a principal ideal $\langle f \rangle$ in R_d we abbreviate $\mathsf{V}(\langle f \rangle)$ to $\mathsf{V}(f)$.

Example 5.1. (One variable, principal ideal) Let $0 \neq f(u) \in R_1$, which we assume is not a unit in R_1 (i.e. not ± 1 times a monomial). Multiplying $f(u)$ by a monomial if necessary, we can also assume that $f(u) = c_r u^r + c_{r-1}u^{r-1} + \cdots + c_1 u + c_0$, where $c_j \in \mathbb{Z}$ and $c_r c_0 \neq 0$. The unit 'sphere' in $\mathbb{R}$ is $\mathsf{S}_0 = \{1, -1\}$. By Lemma 4.4, $1 \in \mathsf{N}^{\mathsf{n}}(\alpha_{R_1/\langle f \rangle})$ if and only if $|c_r| > 1$. If $|c_r| = 1$, then $f(u) = \pm \prod_{j=1}^r (u - \lambda_j)$, so that $\prod_{j=1}^r |\lambda_j| = |c_0| \geq 1$. Hence $a = \max_{1 \leq j \leq r} |\lambda_j| \geq 1$. If $a = 1$, then $\alpha_{R_1/\langle f \rangle}$ is not expansive, and so trivially $1 \in \mathsf{N}(\alpha_{R_1/\langle f \rangle})$. If $a > 1$, then $[0, \infty) \cdot 1 \cap \log|\mathsf{V}(f)| \neq \varnothing$, and so $1 \in \mathsf{N}^{\mathsf{v}}(\alpha_{R_1/\langle f \rangle})$. In all cases we conclude that $1 \in \mathsf{N}(\alpha_{R_1/\langle f \rangle})$. Similarly, $-1 \in \mathsf{N}(\alpha_{R_1/\langle f \rangle})$. Thus $\mathsf{N}(\alpha_{R_1/\langle f \rangle}) = \mathsf{S}_0$.

The following polynomials illustrate some possible combinations of N^{n} and N^{v}:

(a) $f(u) = u^2 - u - 1$, $\mathsf{N}^{\mathsf{n}} = \varnothing$ and $\mathsf{N}^{\mathsf{v}} = \{1, -1\}$;
(b) $f(u) = u - 2$, $\mathsf{N}^{\mathsf{n}} = \{-1\}$ and $\mathsf{N}^{\mathsf{v}} = \{1\}$;
(c) $f(u) = 2$, $\mathsf{N}^{\mathsf{n}} = \{1, -1\}$ and $\mathsf{N}^{\mathsf{v}} = \varnothing$;
(d) $f(u) = 2u^2 - 6u + 3$, $\mathsf{N}^{\mathsf{n}} = \{1, -1\}$ and $\mathsf{N}^{\mathsf{v}} = \{1, -1\}$.

Remark 5.2. A result going back to the PhD thesis of Schwartzman shows that there are no 'one-sided expansive homeomorphisms' except on finite spaces (see [**BL**, Theorem 3.9] for a discussion). From this it follows that if α is an algebraic $\mathbb{Z}$-action on an infinite group, then $\mathsf{N}(\alpha) = \mathsf{S}_0$, providing an alternative approach to Example 5.1.

Example 5.3. (Two variables, principal ideal) Let $f = \sum c_f(\mathbf{n}) u^{\mathbf{n}} \in R_2$. The Newton polyhedron $\mathcal{N}(f)$ of f is the convex hull of $\{\mathbf{n} \in \mathbb{Z}^2 : c_f(\mathbf{n}) \neq 0\}$. If $\mathcal{N}(f)$ is a point or line segment, then we are essentially reduced to Example 5.1, so we assume here that $\mathcal{N}(f)$ is 2-dimensional.

List the vertices of $\mathcal{N}(f)$ as $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_r$, so that the line segment $[\mathbf{n}_j, \mathbf{n}_{j+1}]$ is an edge of $\mathcal{N}(f)$ (with the convention that $\mathbf{n}_{r+1} = \mathbf{n}_1$). Let $\mathbf{v}_j \in \mathsf{S}_1$ denote the outward unit normal vector to $[\mathbf{n}_j, \mathbf{n}_{j+1}]$. If A_j denotes the open arc from $\mathbf{v}_{j-1}$ to $\mathbf{v}_j$, then S_1 is subdivided into the points $\mathbf{v}_j$ and arcs A_j (see Figure 1). This subdivision of S_1 represents the 'spherical dual polygon' to $\mathcal{N}(f)$, with vertices $\mathbf{n}_j$ of $\mathcal{N}(f)$ corresponding to edges A_j, and edges $[\mathbf{n}_j, \mathbf{n}_{j+1}]$ in $\mathcal{N}(f)$ corresponding to vertices $\mathbf{v}_j$.

By Lemma 4.4, each $\mathbf{v}_j \in \mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\langle f \rangle})$. If $|c_f(\mathbf{n}_j)| > 1$, we also have that $A_j \subset \mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\langle f \rangle})$. If $|c_f(\mathbf{n}_j)| = 1$, then an argument using $\mathsf{V}(f)$ similar to that in Example 5.1 shows that $A_j \subset \mathsf{N}^{\mathsf{v}}(\alpha_{R_2/\langle f \rangle})$. Hence in every case $A_j \subset \mathsf{N}(\alpha_{R_2/\langle f \rangle})$, so that $\mathsf{N}(\alpha_{R_2/\langle f \rangle}) = \mathsf{S}_1$ (see [**Sc**] for a detailed treatment of this situation).

We examine two specific polynomials. To describe sets in S_1 we use the notation $\mathbf{v}_\theta = (\cos\theta, \sin\theta)$.

(a) Let $f(u, v) = 3 + u + v$. Then $\mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\langle f \rangle}) = \{\mathbf{v}_\theta : \pi \leq \theta \leq 3\pi/2\} \cup \{\mathbf{v}_{\pi/4}\}$. The set $\log|\mathsf{V}(f)|$ is depicted in Figure 2(a), where the boundary curves are parameterized by $(\log r, \log|3 \pm r|)$ for $0 < r < \infty$. Projecting this set radially to S_1 shows that $\mathsf{N}^{\mathsf{v}}(\alpha_{R_2/\langle f \rangle}) = \{\mathbf{v}_\theta : -\pi/2 < \theta < \pi\}$, and so $\mathsf{N}(\alpha_{R_2/\langle f \rangle}) = \mathsf{S}_1$.

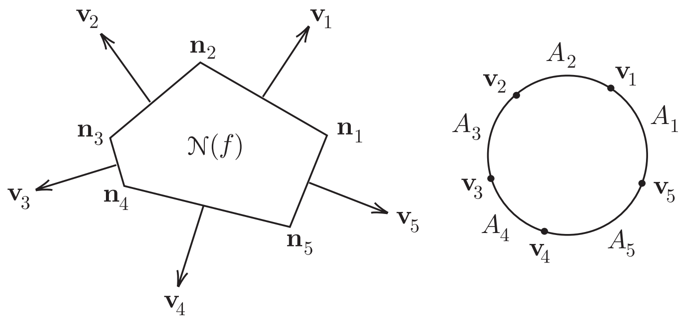


FIGURE 1. Newton polygon and its spherical dual.

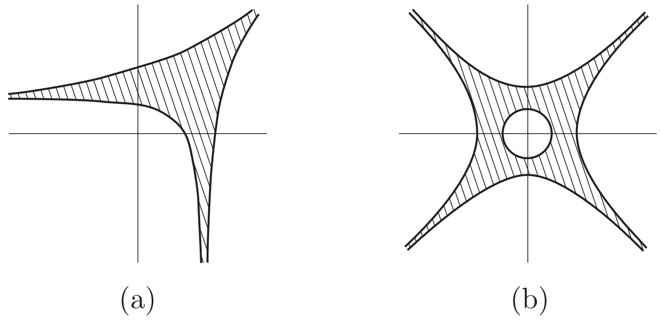


FIGURE 2. Logarithmic images of two varieties.

(b) Let $f(u,v) = 5 + u + u^{-1} + v + v^{-1}$. Then $\mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\langle f\rangle})$ consists of just the four points on S_1 corresponding to the four outward unit normals of $\mathcal{N}(f)$. However, $\log|\mathsf{V}(f)|$, shown in Figure 2(b), covers all directions, so that $\mathsf{N}^{\mathsf{v}}(\alpha_{R_2/\langle f\rangle}) = \mathsf{S}_1$.

Example 5.4. (Three variables, principal ideal) Let $f = \sum c_f(\mathbf{n})u^{\mathbf{n}} \in R_3$, and $\mathcal{N}(f)$ denote the Newton polyhedron of f. If $\mathcal{N}(f)$ has dimension less than or equal to 2, then we are reduced to either Example 5.1 or Example 5.3, so we will assume that $\mathcal{N}(f)$ is 3-dimensional.

We form the 'spherical dual polytope' to $\mathcal{N}(f)$ on S_2 as follows. For each 2-dimensional face F of $\mathcal{N}(f)$ let $\mathbf{v}_F$ be the outward unit normal to F. If faces F and F' share a common edge e, draw a great circle arc A_e from $\mathbf{v}_F$ to $\mathbf{v}_{F'}$. These arcs subdivide S_2 into open regions $B_{\mathbf{n}}$ corresponding to vertices $\mathbf{n}$ of $\mathcal{N}(f)$ (see Figure 3).

By Lemma 4.4, each arc $A_e \subset \mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\langle f\rangle})$. If $\mathbf{n}$ is a vertex of $\mathcal{N}(f)$ with $|c_f(\mathbf{n})| > 1$, then also $B_{\mathbf{n}} \subset \mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\langle f\rangle})$. If $|c_f(\mathbf{n})| = 1$, then $B_{\mathbf{n}} \cap \mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\langle f\rangle}) = \varnothing$, but an analysis similar to the previous examples shows that $B_{\mathbf{n}} \subset \mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\langle f\rangle})$. Again we find that $\mathsf{N}(\alpha_{R_3/\langle f\rangle})$ fills out the entire sphere S_2.

For the principal ideals considered so far we found that the non-expansive set is always the whole sphere. This is true in general.

PROPOSITION 5.5. *Let $f \in R_d$ generate a proper ideal. Then* $\mathsf{N}(\alpha_{R_d/\langle f\rangle}) = \mathsf{S}_{d-1}$.

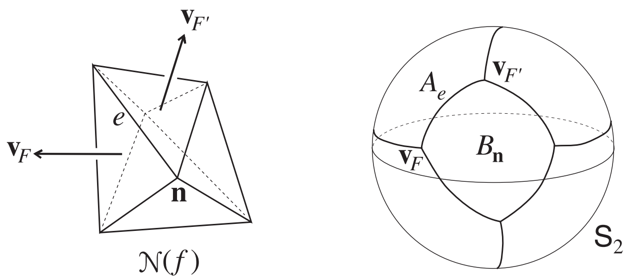


FIGURE 3. The Newton polytope and its spherical dual.

Proof. We could proceed along the lines of the previous examples, which is direct but technically complicated. However, there is a short proof using the notion of a homoclinic point (which is treated in detail in §9).

If $\mathsf{V}(f) \cap \mathbb{S}^d \neq \varnothing$, then $\alpha_{R_d/\langle f\rangle}$ is not expansive by Theorem 3.1, so that $\mathsf{N}(\alpha_{R_d/\langle f\rangle}) = \mathsf{S}_{d-1}$ and we are done. So we may suppose that $\mathsf{V}(f) \cap \mathbb{S}^d = \varnothing$. We represent $X_{R_d/\langle f\rangle}$ as $\{x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^d\}$, so that $\alpha_{R_d/\langle f\rangle}$ is the shift-action on $X_{R_d/\langle f\rangle}$. For $t \in \mathbb{T}$ let $|t| = \min\{|t-n| : n \in \mathbb{Z}\}$. A point $x \in X_{R_d/\langle f\rangle}$ is *homoclinic* if $|x_\mathbf{n}| \to 0$ as $\|\mathbf{n}\| \to \infty$. Consider the function F on $\mathbb{S}^d$ defined by $F(s_1, \dots, s_d) = f(s_1^{-1}, \dots, s_d^{-1})$. Then F does not vanish on $\mathbb{S}^d$ by our assumption on $\mathsf{V}(f)$. Let $(1/F)\widehat{\ }(\mathbf{n})$ be the Fourier transform of $1/F$ at $\mathbf{n} \in \mathbb{Z}^d$. Define $x^\triangle$ by setting $x^\triangle_\mathbf{n}$ to be the reduction mod 1 of $(1/F)\widehat{\ }(\mathbf{n})$. By [**LS**, Lemma 4.5], $x^\triangle \in X_{R_d/\langle f\rangle}$, and $|x^\triangle_\mathbf{n}| \to 0$ as $\|\mathbf{n}\| \to \infty$ by the Riemann–Lebesgue lemma. Hence $x^\triangle$ is a homoclinic point.

Let $\mathbf{v} \in \mathsf{S}_{d-1}$ be arbitrary. Let $H = H_\mathbf{v}$ and let $\varepsilon > 0$. Choose r so that $|x^\triangle_\mathbf{n}| < \varepsilon$ for $\|\mathbf{n}\| > r$. Pick $\mathbf{k} \in \mathbb{Z}^d$ such that $\operatorname{dist}(\mathbf{k}, H) > r$, and put $y = \alpha^\mathbf{k}_{R_d/\langle f\rangle}(x^\triangle)$. Then $|y_\mathbf{n}| < \varepsilon$ for all $\mathbf{n} \in H_\mathbb{Z}$. Since ε was arbitrary, $\mathbf{v} \in \mathsf{N}(\alpha_{R_d/\langle f\rangle})$ for every $\mathbf{v} \in \mathsf{S}_{d-1}$. □

We remark that a proof of the unoriented version of the previous result, namely that $\mathsf{N}_{d-1}(\alpha_{R_d/\langle f\rangle}) = \mathsf{G}_{d-1}$, can be given using entropy. For if $\mathsf{h}(\alpha_{R_d/\langle f\rangle}) = 0$, then f is a product of generalized cyclotomic polynomials, and then $\alpha_{R_d/\langle f\rangle}$ is not expansive. If $\mathsf{h}(\alpha_{R_d/\langle f\rangle}) > 0$, then the entropy of $\alpha_{R_d/\langle f\rangle}$ along every $(d-1)$-plane is infinite, and so no $(d-1)$-plane can be expansive.

For non-principal prime ideals the non-expansive set can exhibit more variety.

Example 5.6. (Ledrappier's example) We revisit Example 2.5. Let $\mathfrak{p} = \langle 2, 1+u+v\rangle$, which is easily seen to be a prime ideal in R_2. Then $\mathsf{V}(\mathfrak{p}) = \varnothing$, so that $\mathsf{N}^\mathsf{v}(\alpha_{R_2/\mathfrak{p}}) = \varnothing$. On the other hand, the outward normals to the sides of the Newton polygon $\mathcal{N}(1+u+v)$ are non-expansive, and by Lemma 4.4 these are the only non-expansive vectors. Thus $\mathsf{N}(\alpha_{R_2/\mathfrak{p}}) = \{\mathbf{v}_{\pi/4}, \mathbf{v}_\pi, \mathbf{v}_{3\pi/2}\}$.

Example 5.7. (3-dimensional Ledrappier example) Let $\mathfrak{p} = \langle 2, 1+u+v+w\rangle$, again easily seen to be a prime ideal in R_3. Then $\mathsf{V}(\mathfrak{p}) = \varnothing$, so that $\mathsf{N}^\mathsf{v}(\alpha_{R_3/\mathfrak{p}}) = \varnothing$. The non-expansive set was determined in [**BL**, Example 2.9] to be the 1-skeleton of the spherical dual to the Newton polytope $\mathcal{N}(1+u+v+w)$. This is depicted in Figure 4, where $\mathsf{N}^\mathsf{n}(\alpha_{R_3/\mathfrak{p}})$ consists of the six great circle arcs determined by the six edges of $\mathcal{N}(1+u+v+w)$.

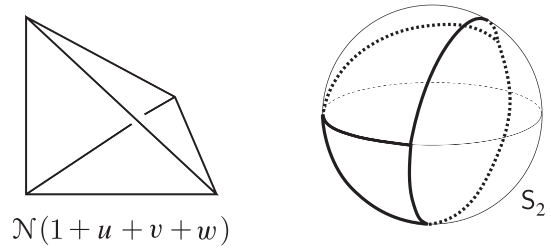


FIGURE 4. The non-expansive set for 3-dimensional Ledrappier.

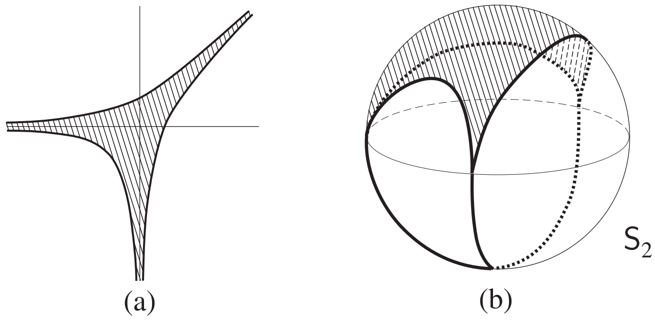


FIGURE 5. Non-expansive set for Example 5.8.

Example 5.8. (Non-expansive set with non-empty interior) Let $d = 3$ and $\mathfrak{p} = \langle 1 + u + v, w - 2\rangle$. We first prove that $\mathfrak{p}$ is a prime ideal in R_3.

Define $\phi\colon R_3 \to \mathbb{Z}[t, 1/2t(t+1)]$ by $\phi(f) = f(t, -t-1, 2)$. Clearly $\mathfrak{p} \subset \ker\phi$. Observe that $\mathbb{Z}[t, 1/2t(t+1)]$ is a subring of $\mathbb{Q}(t)$, hence an integral domain. Define $\psi\colon \mathbb{Z}[t] \to R_3/\mathfrak{p}$ by $\psi(t) = u + \mathfrak{p}$. Note that $\psi(2t(t+1)) = 2u(u+1) + \mathfrak{p} = -uvw + \mathfrak{p}$, a unit in $R_3/\mathfrak{p}$. Hence ϕ extends uniquely to $\mathbb{Z}[t, 1/2t(t+1)]$, and this extension is therefore the inverse of the map $R_3/\mathfrak{p} \to \mathbb{Z}[t, 1/2t(t+1)]$ induced by ϕ. Hence $\mathfrak{p} = \ker\phi$. Thus $R_3/\mathfrak{p}$ is isomorphic to the integral domain $\mathbb{Z}[t, 1/2t(t+1)]$, so $\mathfrak{p}$ is prime. (We are grateful to Paul Smith for showing us this point of view.)

Next we determine $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}})$. Since $\mathsf{V}(\mathfrak{p}) = \{(z, -z-1, 2) : z \in \mathbb{C}\}$, we see that $\log|\mathsf{V}(\mathfrak{p})|$ lies in a plane at height $\log 2$ above the origin, and in this plane it has the shape shown in Figure 5(a), where the boundary curves are parameterized by $(\log r, \log|r \pm 1|)$ for $0 < r < \infty$. When projected radially to S_2, we obtain the set in the upper hemisphere shown in Figure 5(b), with three cusps on the equator.

Finally, let us compute $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$. Using Lemma 4.4, the polynomial $w - 2 \in \mathfrak{p}$ shows that the open upper hemisphere in S_2 is disjoint from $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$. Furthermore, $1+u+v \in \mathfrak{p}$ shows that no points in the lower hemisphere are in $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$ either, with the possible exceptions of those on the three quarter meridians shown in Figure 5(b). We will show that each of these quarter meridians is contained in $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$, so that they, combined with $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}})$ in the upper hemisphere, comprise all of $\mathsf{N}(\alpha_{R_3/\mathfrak{p}})$.

We will treat the meridian from $(0,-1,0)$ to $(0,0,-1)$, the other two being similar. Since $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$ is closed by Remark 4.5, it is enough to show that unit vectors in the directions $(0,-a,-b)$ are non-Noetherian, where a and b are positive integers. Let $H \in \mathsf{H}_3$ be $\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot (0,a,b) \geq 0\}$. Using the isomorphism $\phi\colon R_3/\mathfrak{p} \to \mathbb{Z}[t, 1/2t(t+1)]$ above, the subring R_H is mapped to $\mathbb{Z}[t^{\pm 1}, (-t-1)^m 2^n : am+bn \geq 0]$. Then $R_3/\mathfrak{p}$ is Noetherian over R_H if and only if $\mathbb{Z}[t, 1/2t(t+1)]$ is finitely generated over $\mathbb{Z}[t^{\pm 1}, (-t-1)^m 2^n : am+bn \geq 0]$. By Lemma 4.4, this is clearly equivalent to whether we can write 1 as a combination, using coefficients in $\mathbb{Z}[t^{\pm 1}]$, of expressions of the form $(-t-1)^m 2^n$, where $am+bn > 0$.

Suppose this to be the case, so that

$$1 = \sum_{(m,n)\in F} f_{mn}(t)(-t-1)^m 2^n, \tag{5.1}$$

where $f_{mn}(t) \in \mathbb{Z}[t^{\pm 1}]$ and F is a finite set of $(m,n) \in \mathbb{Z}^2$ for which $am+bn > 0$. Let $|\cdot|_2$ denote the extension of the 2-adic norm on $\mathbb{Q}$ to $\mathbb{Q}(2^{1/b})$. Substitute $t = 2^{a/b} - 1$ in (5.1). Since $|2^{a/b}-1|_2 = 1$, it follows that $|f_{mn}(2^{a/b}-1)|_2 \leq 1$. Hence

$$\begin{aligned} 1 = |1|_2 &= \left| \sum_{(m,n)\in F} f_{mn}(2^{a/b}-1)(-2^{a/b})^m 2^n \right|_2 \\ &\leq \max_{(m,n)\in F} |f_{mn}(2^{a/b}-1)|_2| - 2^{a/b}|_2^m |2|_2^n \\ &\leq \max_{(m,n)\in F} 2^{-(am+bn)/b} < 1. \end{aligned}$$

This contradiction shows that (5.1) is impossible, so that each rational direction $(0,-a,-b)$ is non-Noetherian. See also Example 8.6 for another approach.

Remark 5.9. In the previous example, and in many others, the non-Noetherian and variety parts of the non-expansive set are 'glued together' along a set which can be described by the asymptotic behavior of the logarithmic image of the variety. For instance, in Example 5.8, the variety part $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}})$ in the upper hemisphere and the non-Noetherian part $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$ in the lower hemisphere are glued together at the three cusp points on the equator, which are also the three asymptotic directions of $\log|\mathsf{V}(\mathfrak{p})|$. This illustrates a general phenomenon of the *logarithmic limit set of an algebraic variety*, introduced by Bergman [**B**]. He showed that this set is always contained in a finite union of lower-dimensional great spheres. We point out that this set has an alternative description as the set of half-spaces H for which $(R_d/\mathfrak{p}) \otimes \mathbb{Q}$ is not Noetherian over $R_H \otimes \mathbb{Q}$, so in this sense is also a limiting part of the non-Noetherian set (we are grateful to Bernd Sturmfels for pointing this out to us).

6. *Further analysis of non-expansive sets*

Let M be an R_d-module, which we will assume throughout this section to be Noetherian. In §4 we showed that

$$\mathsf{N}(\alpha_M) = \bigcup_{\mathfrak{p}\in \operatorname{asc}(M)} \mathsf{N}(\alpha_{R_d/\mathfrak{p}}), \tag{6.1}$$

where

$$\mathsf{N}(\alpha_{R_d/\mathfrak{p}}) = \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}}) \cup \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}). \tag{6.2}$$

In this section we refine our analysis of $\mathsf{N}(\alpha_M)$, showing that only isolated primes in $\operatorname{asc}(M)$ are relevant. We provide an algorithm using Gröbner bases to compute $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{a}})$ for ideals $\mathfrak{a}$ in R_d, and show how this can be used to compute $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ using the Fitting ideal of M. The question of when either $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ or $\mathsf{N}^{\mathsf{v}}(\alpha_M)$ is empty is answered in Proposition 6.9.

We begin by extending the definitions of N^{n} and N^{v}.

Definition 6.1. For M a Noetherian R_d-module, define

$$\mathsf{N}^{\mathsf{n}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\operatorname{asc}(M)} \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}}) \quad \text{and} \quad \mathsf{N}^{\mathsf{v}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\operatorname{asc}(M)} \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}). \tag{6.3}$$

Hence by (6.1) and (6.2),

$$\mathsf{N}(\alpha_M) = \mathsf{N}^{\mathsf{n}}(\alpha_M) \cup \mathsf{N}^{\mathsf{v}}(\alpha_M). \tag{6.4}$$

Remark 6.2. Alternatively, we could have defined $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ to be $\{H \in \mathsf{H}_d : M \text{ is not } R_H\text{-Noetherian}\}$. For M has a prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with $M_j/M_{j-1} \cong R_d/\mathfrak{q}_j$, where $\mathfrak{q}_j$ is a prime ideal containing some $\mathfrak{p}_j \in \operatorname{asc}(M)$, and furthermore every $\mathfrak{p} \in \operatorname{asc}(M)$ occurs as some $\mathfrak{q}_j$ [**E**, p. 93]. Then M fails to be R_H-Noetherian if and only if some quotient M_j/M_{j-1} is not R_H-Noetherian, establishing the equivalence of the two definitions of $\mathsf{N}^{\mathsf{n}}(\alpha_M)$. Note that Remark 4.5 shows that $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ is always closed.

Order $\operatorname{asc}(M)$ by inclusion, and let $\min(M)$ denote the subset of minimal elements of $\operatorname{asc}(M)$. Members of $\min(M)$ are called the *isolated primes* for M. They play an essential role in the primary decomposition of M, and also govern the expansive subdynamics of α_M.

PROPOSITION 6.3. *Let M be a Noetherian R_d-module. Then*

$$\mathsf{N}^{\mathsf{n}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\min(M)} \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}}), \quad \mathsf{N}^{\mathsf{v}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\min(M)} \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}), \tag{6.5}$$

and so

$$\mathsf{N}(\alpha_M) = \bigcup_{\mathfrak{p}\in\min(M)} \mathsf{N}(\alpha_{R_d/\mathfrak{p}}).$$

Proof. Let $\mathfrak{q} \in \operatorname{asc}(M)$ and choose $\mathfrak{p} \in \min(M)$ with $\mathfrak{p} \subset \mathfrak{q}$. The natural surjection $R_d/\mathfrak{p} \to R_d/\mathfrak{q}$ shows that, for every half-space $H \in \mathsf{H}_d$, if $R_d/\mathfrak{q}$ is not R_H-Noetherian, then $R_d/\mathfrak{p}$ is not R_H-Noetherian. Hence $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{q}}) \subset \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$, establishing the first equality in (6.5). For the second equality, observe that if $\mathfrak{p} \subset \mathfrak{q}$, then $\log|\mathsf{V}(\mathfrak{q})| \subset \log|\mathsf{V}(\mathfrak{p})|$. $\square$

By (6.3) and (6.4), we can compute $\mathsf{N}^{\mathsf{v}}(\alpha_M)$ as the union over all $\mathfrak{p} \in \operatorname{asc}(M)$ of the radial projections of $\log|\mathsf{V}(\mathfrak{p})|$ to S_{d-1}. We next give two approaches to computing $\mathsf{N}^{\mathsf{n}}(\alpha_M)$. We first introduce some convenient terminology.

Definition 6.4. Let $f \in R_d$ and $H = H_{\mathbf{v}} \in \mathsf{H}_d$. The *support of* f is the set $\{\mathbf{n} \in \mathbb{Z}^d : c_f(\mathbf{n}) \neq 0\}$. Say that f has an *H-exposed vertex* $\mathbf{n}$ if $c_f(\mathbf{n}) \neq 0$ and $\mathbf{m} \cdot \mathbf{v} < \mathbf{n} \cdot \mathbf{v}$ for every $\mathbf{m} \in \operatorname{supp}(f) \smallsetminus \{\mathbf{n}\}$. If f has an H-exposed vertex $\mathbf{n}$ with $c_f(\mathbf{n}) = 1$, then f is called *H-monic*.

Note that Lemma 4.4 shows that M is R_H-Noetherian if and only if there is an H-monic polynomial that annihilates M. In particular, if $\mathfrak{a}$ is an ideal in R_d, then $R_d/\mathfrak{a}$ is R_H-Noetherian if and only if $\mathfrak{a}$ contains an H-monic polynomial. Observe that any factor of an H-monic polynomial is ± 1 times an H-monic polynomial.

We begin with principal ideals $\mathfrak{a} = \langle f \rangle$. By the preceding paragraph, $R_d/\langle f \rangle$ is R_H-Noetherian if and only if f is H-monic. An obvious extension of terminology in Example 5.4 then shows that $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\langle f \rangle})$ is the $(d-2)$-skeleton of the spherical dual of the Newton polyhedron $\mathcal{N}(f)$ of f together with those $(d-1)$-faces of the spherical dual corresponding to vertices $\mathbf{n}$ of $\mathcal{N}(f)$ for which $|c_f(\mathbf{n})| > 1$.

Now suppose that $\mathfrak{a} = \langle f_1, \dots, f_r \rangle \subset R_d$. For $H \in \mathsf{H}_d$ it is tempting, but wrong, to believe that $\mathfrak{a}$ contains an H-monic polynomial if and only if one of the f_j is H-monic.

Example 6.5. Let $d = 2$, $f = u - 2$, $g = v - 3$, and $\mathfrak{a} = \langle f, g \rangle$. Take $\mathbf{v} = (-1/\sqrt{2}, -1/\sqrt{2}) \in \mathsf{S}_1$ and $H = H_{\mathbf{v}}$. Then neither f nor g is H-monic. However, $f - g = u - v + 1 \in \mathfrak{a}$ is H-monic.

Here $\mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\langle f \rangle}) = \{\mathbf{v} \in \mathsf{S}_1 : \mathbf{v} \cdot \mathbf{e}_1 \leq 0\}$ and $\mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\langle g \rangle}) = \{\mathbf{v} \in \mathsf{S}_1 : \mathbf{v} \cdot \mathbf{e}_2 \leq 0\}$, and so the intersection of these is the quarter-circle of S_1 in the third quadrant. However,

$$\{-\mathbf{e}_1, -\mathbf{e}_2\} = \mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\mathfrak{a}}) \subsetneqq \mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\langle f \rangle}) \cap \mathsf{N}^{\mathsf{n}}(\alpha_{R_2/\langle g \rangle}).$$

Given an ideal $\mathfrak{a}$ in R_d, we would like to compute a finite generating set F for $\mathfrak{a}$ with the property that if $H \in \mathsf{H}_d$ then $\mathfrak{a}$ contains an H-monic polynomial if and only if F contains an H-monic polynomial. For then

$$\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{a}}) = \bigcap_{f \in F} \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\langle f \rangle}),$$

where each term in the intersection has the description given above.

We compute such a set F using the theory of Gröbner bases [**AL**] and universal Gröbner bases [**St**], which we now briefly summarize. Let $\mathbb{N} = \{0, 1, 2, \dots\}$. The monomials in $\mathbb{Z}[u_1, \dots, u_d]$ correspond to elements $\mathbf{n} \in \mathbb{N}^d$ via $u^{\mathbf{n}} \leftrightarrow \mathbf{n}$. A *term order* $\prec$ on $\mathbb{Z}[u_1, \dots, u_d]$ is a total order on $\mathbb{N}^d$ such that (1) $\mathbf{0} \prec \mathbf{k}$ for every $\mathbf{0} \neq \mathbf{k} \in \mathbb{N}^d$, and (2) $\mathbf{k} \prec \mathbf{m}$ implies that $\mathbf{k} + \mathbf{n} \prec \mathbf{m} + \mathbf{n}$ for every $\mathbf{k}, \mathbf{m}, \mathbf{n} \in \mathbb{N}^d$. Fix a term order $\prec$. For $0 \neq f \in \mathbb{Z}[u_1, \dots, u_d]$ let $\operatorname{lt}_{\prec}(f)$ denote the unique *leading term* $c_f(\mathbf{n})u^{\mathbf{n}}$ of f, where $\mathbf{n}$ is maximal with respect to $\prec$ such that $c_f(\mathbf{n}) \neq 0$. Let $\mathfrak{b}$ be an ideal of $\mathbb{Z}[u_1, \dots, u_d]$. Then a finite set $G \subset \mathfrak{a}$ is a *Gröbner basis for* $\mathfrak{b}$ *with respect to* $\prec$ if for every $f \in \mathfrak{b}$ there is a $g \in G$ for which $\operatorname{lt}_{\prec}(g)$ divides $\operatorname{lt}_{\prec}(f)$ in $\mathbb{Z}[u_1, \dots, u_d]$.

Although there are infinitely many term orders on $\mathbb{Z}[u_1, \dots, u_d]$, it is shown in [**St**, pp. 1–2] that every ideal has a *universal Gröbner basis*, i.e. a finite set that is a Gröbner basis with respect to *every* term order. Although the proof in [**St**] is for the case when the coefficients lie in a field, it is easy to adapt the argument to integer coefficients. Also, [**St**] provides effective algorithms for computing a universal Gröbner basis.

PROPOSITION 6.6. *Let $\mathfrak{a}$ be an ideal in R_d. Then there is an effectively computable finite generating set $F \subset \mathfrak{a}$ such that*

$$\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{a}}) = \bigcap_{f \in F} \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\langle f\rangle}). \tag{6.6}$$

Proof. The theory of Gröbner bases applies to polynomials rather than Laurent polynomials. In order to apply it here, we will subdivide $\mathbb{Z}^d$ into the 2^d orthants corresponding to the signs of the entries, and use a universal Gröbner basis in each of these orthants.

Let $D = \{1, 2, \dots, d\}$. For each subset $E \subset D$ define $E(j) = 1$ if $j \in E$ and $E(j) = -1$ if $j \notin E$. Let $S_E = \mathbb{Z}[u_1^{E(1)}, \dots, u_d^{E(d)}]$, and put $\mathfrak{a}_E = \mathfrak{a} \cap S_E$, an ideal in S_E. For each E we will construct a finite set $F_E \subset \mathfrak{a}_E$, and show that $F = \bigcup_{E \subset D} F_E \subset \mathfrak{a}$ satisfies (6.6).

For notational simplicity we give the construction for $E = D$, and indicate the modifications needed for general E. Here $S_D = \mathbb{Z}[u_1, \dots, u_d]$. Choose a finite set $\{f_1, \dots, f_r\}$ of generators for $\mathfrak{a}_D$ over S_D. Let $R = S_D[t] = \mathbb{Z}[u_1, \dots, u_d, t]$ and define

$$\mathfrak{b}_D = \langle f_1, \dots, f_r, t - u_1 u_2 \dots u_d\rangle \subset R$$

(for general E the last generator is $t - \prod_j u_j^{E(j)}$). Construct a universal Gröbner basis G_D for $\mathfrak{b}_D$. Define $\phi\colon R \to R_d$ by $\phi(u_j) = u_j$ and $\phi(t) = u_1 u_2 \dots u_d$ (for general E let $\phi(t) = \prod_j u_j^{E(j)}$). Clearly $\phi(\mathfrak{b}_D) = \mathfrak{a}_D$. Define $F_D = \phi(G_D)$.

We show that if $\mathbf{v} \geq 0$ and $H = H_{\mathbf{v}}$, then there is an H-monic polynomial f in $\mathfrak{a}$ if and only if some element in F_D is also H-monic. (For general vectors $\mathbf{v}$, use the above construction for $E = \{j : v_j \geq 0\}$.) First, multiply f by a monomial so that $f \in \mathfrak{a}_D$, and also that the H-exposed vertex of f has the form $N\mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1)$. Hence $f = u^{N\mathbf{1}} + h(u)$, where

$$\mathbf{n} \cdot \mathbf{v} < (N\mathbf{1}) \cdot \mathbf{v} \quad \text{for all } \mathbf{n} \in \operatorname{supp}(h). \tag{6.7}$$

Therefore $t^N + h(u) \in \mathfrak{b}_D$.

Define a term order on monomials in $R = S_D[t]$ by declaring that $(\mathbf{k}, K) \prec (\mathbf{m}, M)$ if and only if $(\mathbf{k} + K\mathbf{1}) \cdot \mathbf{v} \leq (\mathbf{m} + M\mathbf{1}) \cdot \mathbf{v}$, and in case of equality $K \geq M$, and in case of equality there, that $\mathbf{k}$ is lexicographically less than $\mathbf{m}$. Then $\operatorname{lt}_\prec(t^N + h(u)) = t^N$ by (6.7). Since G_D is a Gröbner basis for $\prec$, there is a $g \in G_D$ with $\operatorname{lt}_\prec(g)$ dividing t^N. Hence $\operatorname{lt}_\prec(g) = t^M$ for some $M \leq N$. Let $\phi(g) = u^{\mathbf{m}} + k(u)$ where $\phi(t^M) = u^{\mathbf{m}}$. We claim that $\phi(g)$ is a polynomial in F_D that is H-monic. This follows since every monomial in g that could give rise to a term in $\phi(g)$ strictly larger than m would have to already be greater than t^M with respect to $\prec$. □

To compute $\mathsf{N}(\alpha_M)$ for a general Noetherian R_d-module M, we use the notion of Fitting ideal $\mathfrak{f}(M)$ to reduce to the situation of Proposition 6.6 (see [**E**, Ch. 20] or [**L**, §XIII.10] for background). To define $\mathfrak{f}(M)$, suppose that M is generated by $m_1, m_2, \dots, m_r$. Let $K \subset R_d^r$ be the kernel of the map $(f_1, \dots, f_r) \mapsto f_1 m_1 + \dots + f_r m_r$. Since K is Noetherian, it is finitely generated over R_d, say by $(a_{11}, \dots, a_{1r}), \dots, (a_{s1}, \dots, a_{sr})$. Let A be the $s \times r$ matrix $[a_{ij}]$, so that $M \cong R_d^r/(R_d^s A)$. Then $\mathfrak{f}(M)$ is defined to be the ideal in R_d generated by all the $r \times r$ subdeterminants of A. This ideal is independent of the presentation of M, and can be effectively computed (see [**AL**]). Also, if $\mathrm{ann}(M) = \{f \in R_d : f \cdot M = 0\}$, then

$$(\mathrm{ann}(M))^r \subset \mathfrak{f}(M) \subset \mathrm{ann}(M) \subset \bigcap_{\mathfrak{p} \in \mathrm{asc}(M)} \mathfrak{p}. \tag{6.8}$$

In [**EW**] it is shown that both entropy and expansiveness of α_M can be computed from $\mathfrak{f}(M)$, although more subtle dynamical information requires higher order Fitting ideals. The next result shows that both pieces of $\mathsf{N}(\alpha_M)$ can be found from $\mathfrak{f}(M)$.

PROPOSITION 6.7. *Let M be a Noetherian R_d-module and $\mathfrak{f}(M)$ denote its Fitting ideal. Then*

$$\mathsf{N}^{\mathsf{n}}(\alpha_M) = \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{f}(M)}), \quad \mathsf{N}^{\mathsf{v}}(\alpha_M) = \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{f}(M)}),$$

and so

$$\mathsf{N}(\alpha_M) = \mathsf{N}(\alpha_{R_d/\mathfrak{f}(M)}).$$

Proof. Suppose that M is generated over R_d by r elements. Let $H \in \mathsf{H}_d$, and suppose that M is R_H-Noetherian. By Lemma 4.4, there is an H-monic polynomial $f \in \mathrm{ann}(M)$. Then (6.8) shows that $f^r \in \mathfrak{f}(M)$, and f^r is clearly H-monic. Hence another application of Lemma 4.4 implies that $R_d/\mathfrak{f}(M)$ is H-Noetherian. Thus $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{f}(M)}) \subset \mathsf{N}^{\mathsf{n}}(\alpha_M)$. Conversely, suppose that $R_d/\mathfrak{f}(M)$ is R_H-Noetherian. Since $\mathfrak{f}(M) \subset \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{asc}(M)$, we see that $R_d/\mathfrak{p}$ is R_H-Noetherian for every $\mathfrak{p} \in \mathrm{asc}(M)$. Hence

$$\mathsf{N}^{\mathsf{n}}(\alpha_M) = \bigcup_{\mathfrak{p} \in \mathrm{asc}(M)} \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}}) \subset \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{f}(M)}),$$

proving that $\mathsf{N}^{\mathsf{n}}(\alpha_M) = \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{f}(M)})$.

For the second equality, observe that since $\mathfrak{f}(M) \subset \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{asc}(M)$,

$$\mathsf{N}^{\mathsf{v}}(\alpha_M) = \bigcup_{\mathfrak{p} \in \mathrm{asc}(M)} \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}) \subset \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{f}(M)}).$$

To prove the reverse inclusion, take $H_{\mathbf{v}} \in \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{f}(M)})$. Then there is a $\mathbf{z} \in \mathsf{V}(\mathfrak{f}(M))$ such that $\log|\mathbf{z}| \in [0, \infty)\mathbf{v}$. Suppose that $\mathbf{z} \notin \mathsf{V}(\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{asc}(M)$. Then there are polynomials $g_{\mathfrak{p}} \in \mathfrak{p}$ with $g_{\mathfrak{p}}(\mathbf{z}) \neq 0$. Put $g = \prod_{\mathfrak{p} \in \mathrm{asc}(M)} g_{\mathfrak{p}}$. Then $g \in \bigcap_{\mathfrak{p} \in \mathrm{asc}(M)} \mathfrak{p}$, and so some power $g^k \in \mathrm{ann}(M)$ (to see this, either use a prime filtration of M, or observe that the radical of $\mathrm{ann}(M)$ is $\bigcap_{\mathfrak{p} \in \mathrm{asc}(M)} \mathfrak{p}$). By (6.8) we see that $(g^k)^r \in \mathfrak{f}(M)$. But this contradicts $g^{kr}(\mathbf{z}) = \prod_{\mathfrak{p}} g_{\mathfrak{p}}^{kr}(\mathbf{z}) \neq 0$. Hence there is a $\mathfrak{p} \in \mathrm{asc}(M)$ such that $H_{\mathbf{v}} \in \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}})$. □

Remark 6.8. The set $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ has been investigated in another context by Bieri and Groves [**BG**] using a valuation-theoretic approach to the extension of characters on fields. More specifically, they show in [**BG**, Thm. 8.1] how $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ can be calculated using such characters. In [**M**] these ideas are used to give a purely valuation-theoretic description of all of $\mathsf{N}(\alpha_M)$.

Next, we characterize when $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ or $\mathsf{N}^{\mathsf{v}}(\alpha_M)$ is empty. Note that the answers depend only on the topological nature of X_M. If $\mathfrak{p}$ is a prime ideal in R_d, we let $\operatorname{char}(R_d/\mathfrak{p})$ denote the characteristic of the integral domain $R_d/\mathfrak{p}$.

PROPOSITION 6.9. *Let M be a Noetherian R_d-module.*

(1) *The following conditions are equivalent:*
  (a) $\mathsf{N}^{\mathsf{n}}(\alpha_M) = \varnothing$;
  (b) *M is finitely generated as an abelian group;*
  (c) *X_M is the direct product of a finite-dimensional torus and a finite abelian group.*

(2) *The following conditions are equivalent:*
  (a) $\mathsf{N}^{\mathsf{v}}(\alpha_M) = \varnothing$;
  (b) *M is a torsion abelian group;*
  (c) *X_M is totally disconnected.*

Proof. (1b) $\Rightarrow$ (1a): Suppose that M is finitely generated as an abelian group. Then M is trivially R_H-Noetherian for every $H \in \mathsf{H}_d$, so that $\mathsf{N}^{\mathsf{n}}(\alpha_M) = \varnothing$ by Remark 6.2.

(1a) $\Rightarrow$ (1b): Our proof uses an algebraic analogue of the proof of Theorem 3.6 in [**BL**].

Since $\mathsf{N}^{\mathsf{n}}(\alpha_M) = \varnothing$, by Lemma 4.4 for every $\mathbf{v} \in \mathsf{S}_{d-1}$ there is a polynomial of the form $1 - f(u) = 1 - \sum_{\mathbf{n}\in F} c_f(\mathbf{n})u^{\mathbf{n}} \in R_{H_{\mathbf{v}}}$ that annihilates M and such that $\mathbf{n} \cdot \mathbf{v} < 0$ for every $\mathbf{n} \in F$. This shows that there are $\varepsilon_{\mathbf{v}} > 0$, $r_{\mathbf{v}} > 0$, and a neighborhood $\mathcal{U}(\mathbf{v})$ in S_{d-1} such that if $\mathbf{p} \in \mathbb{R}^d$ with $\|\mathbf{p}\| > r_{\mathbf{v}}$ and $\mathbf{p}/\|\mathbf{p}\| \in \mathcal{U}(\mathbf{v})$, then $F + \mathbf{p} \subset B(\|\mathbf{p}\| - \varepsilon_{\mathbf{v}})$.

The collection $\{\mathcal{U}(\mathbf{v}) : \mathbf{v} \in \mathsf{S}_{d-1}\}$ is an open cover of S_{d-1}, so by compactness there is a finite subcover $\{\mathcal{U}(\mathbf{v}_1), \dots, \mathcal{U}(\mathbf{v}_n)\}$. Put $\varepsilon = \min\{\varepsilon_{\mathbf{v}_j} : 1 \le j \le n\}$ and $r = \max\{r_{\mathbf{v}_j} : 1 \le j \le n\}$. Let M be generated over R_d by $m_1, \dots, m_k$. For $s > 0$ let M_s denote the abelian group generated by $\{u^{\mathbf{n}} \cdot m_i : 1 \le i \le k, \mathbf{n} \in B(s) \cap \mathbb{Z}^d\}$. We claim that $M_r = M$, so that M is finitely generated as an abelian group. We prove this by showing successively that $M_r = M_{r+\varepsilon} = M_{r+2\varepsilon} = \cdots$, so that $M = \bigcup_{q=0}^{\infty} M_{r+q\varepsilon} = M_r$.

Let $\mathbf{p} \in (B(r+\varepsilon) \smallsetminus B(r)) \cap \mathbb{Z}^d$. Then $\mathbf{p}/\|\mathbf{p}\| \in \mathcal{U}(\mathbf{v}_j)$ for some $1 \le j \le n$. By our construction, there is a polynomial $1 - f(u) = 1 - \sum_{\mathbf{n}\in F} c_f(\mathbf{n})u^{\mathbf{n}}$ that annihilates M, and such that $F + \mathbf{p} \subset B(\|\mathbf{p}\| - \varepsilon_{\mathbf{v}_j}) \subset B(r)$. Hence for $1 \le i \le k$ we see that

$$u^{\mathbf{p}} \cdot m_i = u^{\mathbf{p}} f(u) \cdot m_i = \sum_{\mathbf{n}\in F} c_f(\mathbf{n}) u^{\mathbf{n}+\mathbf{p}} \cdot m_i \in M_r.$$

This shows that $M_{r+\varepsilon} \subset M_r$, and the reverse inclusion is trivial. The same argument applied to $M_{r+\varepsilon}$ shows that $M_{r+\varepsilon} = M_{r+2\varepsilon}$, and so on, completing the proof.

(1b)$\Leftrightarrow$(1c): This equivalence is standard from duality.

(2a) $\Rightarrow$ (2b): By assumption, $\mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}) = \varnothing$ for every $\mathfrak{p} \in \operatorname{asc}(M)$. Hence $\operatorname{char}(R_d/\mathfrak{p}) > 0$ for every $\mathfrak{p} \in \operatorname{asc}(M)$ by Hilbert's Nullstellensatz. There is a prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with $M_j/M_{j-1} \cong R_d/\mathfrak{q}_j$, where $\mathfrak{q}_j$ is a prime ideal containing some $\mathfrak{p}_j \in \operatorname{asc}(M)$. Then the integer $\prod_{j=1}^{r} \operatorname{char}(R_d/\mathfrak{p}_j)$ annihilates M, so that M is a torsion abelian group.

(2b) $\Rightarrow$ (2a): Suppose that M is a torsion abelian group. Let $\mathfrak{p} \in \operatorname{asc}(M)$. Then there is an $m \in M$ with $R_d \cdot m \cong R_d/\mathfrak{p}$. Since $R_d \cdot m$ must also be a torsion abelian group, it follows that $\mathrm{char}(R_d/\mathfrak{p}) > 0$. In particular, $\mathfrak{p}$ contains a non-zero constant, so that $\mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}) = \varnothing$. Hence $\mathsf{N}^{\mathsf{v}}(\alpha_M) = \varnothing$.

(2b)$\Leftrightarrow$(2c): This equivalence is standard from duality. □

7. *Expansive rank, entropy rank, and Krull dimension*

Although a $\mathbb{Z}^d$-action nominally involves d commuting transformations, sometimes its true 'rank' is less. In this section we describe two ways to measure this rank. In what follows h denotes topological entropy.

Definition 7.1. Let β be a topological $\mathbb{Z}^d$-action. Define the *expansive rank* of β to be

$$\mathrm{exprk}(\beta) = \min\{k : \mathsf{E}_k(\beta) \neq \varnothing\},$$

and the *entropy rank* of β to be

$$\mathrm{entrk}(\beta) = \max\{k : \text{there is a rational } k\text{-plane } V \text{ with } \mathsf{h}(\beta, V \cap \mathbb{Z}^d) > 0\}.$$

By convention, if β is not expansive we put $\mathrm{exprk}(\beta) = d + 1$, and if the set defining $\mathrm{entrk}(\beta)$ is empty we put $\mathrm{entrk}(\beta) = 0$.

These ranks attempt to measure, from the viewpoints of expansiveness and of entropy, the maximum number of 'independent transformations' in the action such that the remaining transformations are determined in some sense. As a concrete instance of what we have in mind, consider Ledrappier's Example 2.5. If L is an expansive line, and L^1 denotes the thickening of L by 1, then every element of the action can be written as a function of $\alpha^{\mathbf{n}}$ for finitely many $\mathbf{n} \in L^1 \cap \mathbb{Z}^2$. In this sense α has expansive rank 1.

PROPOSITION 7.2. *Let β be a topological $\mathbb{Z}^d$-action. Then* $\mathrm{entrk}(\beta) \leq \mathrm{exprk}(\beta)$.

Proof. Let $\mathrm{exprk}(\beta) = k$. It follows from [**BL**, Theorem 6.3] that if V is a k-plane, then $\mathsf{h}(\beta, V \cap \mathbb{Z}^d) < \infty$. A standard argument now shows that if W is a rational subspace of dimension $\geq k + 1$ then $\mathsf{h}(\beta, W \cap \mathbb{Z}^d) = 0$. Hence $\mathrm{entrk}(\beta) \leq k$. □

We remark that an alternative proof of this proposition can be obtained by modifying arguments of Shereshevsky [**Sh**]. We also observe that in general the inequality here can be strict, for example a zero entropy subshift has entropy rank zero but expansive rank one.

The building blocks for algebraic $\mathbb{Z}^d$-actions are based on modules of the form $R_d/\mathfrak{p}$ for prime ideals $\mathfrak{p}$. For these we investigate expansive and entropy ranks. The appropriate algebraic version of rank is Krull dimension for rings. Recall that the *Krull dimension* $\mathrm{kdim}\,\mathcal{R}$ of a ring $\mathcal{R}$ is the supremum of the lengths r of all chains $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_r$ of prime ideals in $\mathcal{R}$ (see [**E**, Ch. 8] for the necessary background). For example, $\mathrm{kdim}\,R_d = d + 1$. In [**BL**, Theorem 7.5] it is shown that if $\mathfrak{p}$ is a prime ideal in R_d generated by g elements, then

$$\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}) \geq \mathrm{kdim}(R_d/\mathfrak{p}) - 1 \geq d - g,$$

and if $\mathrm{char}(R_d/\mathfrak{p}) > 0$ then

$$\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}) \geq d - g + 1.$$

Since expansive and entropy ranks in some sense measure dimension, one might heuristically expect $\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}})$, $\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}})$ and $\mathrm{kdim}(R_d/\mathfrak{p})$ to coincide. The best that can be said in this direction is as follows.

PROPOSITION 7.3. *Let $\alpha_{R_d/\mathfrak{p}}$ be an expansive action with zero entropy. Then*

$$\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) = \mathrm{kdim}(R_d/\mathfrak{p}) \leq \mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}).$$

If $\mathrm{char}(R_d/\mathfrak{p}) > 0$ *then all of these quantities are equal.*

Proof. Let $k = \mathrm{exprk}(\alpha_{R_d/\mathfrak{p}})$. Since the expansive set is open, there is a rank k lattice in $\mathbb{Z}^d$ for which the restriction of $\alpha_{R_d/\mathfrak{p}}$ is expansive. By a simple change of variables, we may assume that this lattice is generated by the first k unit vectors. That is, the subaction dual to multiplication by $u_1, \ldots, u_k$ is expansive. We thus consider $R_d/\mathfrak{p}$ as an R_k-module. We claim that the images of the variables $u_1, \ldots, u_k$ in $R_d/\mathfrak{p}$ must satisfy a polynomial relation, and also that each u_j for $j > k$ must be algebraic over $u_1, \ldots, u_k$. For the first claim, observe that if the images of $u_1, \ldots, u_k$ in $R_d/\mathfrak{p}$ do not satisfy any polynomial relation, then the natural map $R_k \to R_d/\mathfrak{p}$ is injective, so that $\{0\}$ is a prime ideal associated to the R_k-module $R_d/\mathfrak{p}$. But by Theorem 3.1 this contradicts the assumption that the restriction of $\alpha_{R_d/\mathfrak{p}}$ to the first k variables is expansive. If some u_j were not algebraic over $u_1, \ldots, u_k$, then $R_d/\mathfrak{p}$ would not be a Noetherian R_k-module, again contradicting the expansiveness assumption. It follows that $\mathrm{kdim}(R_d/\mathfrak{p}) \leq \mathrm{kdim}(R_k) - 1 = k$.

Now let $\mathrm{kdim}(R_d/\mathfrak{p}) = k$, and assume first that $\mathrm{char}(R_d/\mathfrak{p}) = 0$. Then any $k+1$ monomials must satisfy two coprime irreducible polynomial relations, so that the entropy of the corresponding $\mathbb{Z}^{k+1}$-action is zero. Hence $\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) \leq \mathrm{kdim}(R_d/\mathfrak{p})$. Now assume without loss of generality that the images of the variables $u_{k+1}, \ldots, u_d$ in $R_d/\mathfrak{p}$ are algebraic over the images of $u_1, \ldots, u_k$, and that $u_1, \ldots, u_k$ satisfy only one polynomial relation. If this relation is not a generalized cyclotomic polynomial, then the corresponding $\mathbb{Z}^k$-subaction has positive entropy [**LSW**, Example 5.4], so that $\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) \geq \mathrm{kdim}(R_d/\mathfrak{p})$. So we may assume that this single polynomial relation is an irreducible generalized cyclotomic polynomial. After a suitable change of variables, this implies that $u_k^r = 1$ in $R_d/\mathfrak{p}$ for some $r \geq 1$. Since the original system is expansive, we may find among the variables $u_{k+1}, \ldots, u_d$ a variable u_j which is not a root of unity. Now the same argument may be applied to the set of variables $u_1, \ldots, u_{k-1}, u_j$. Continuing, we either arrive at a $\mathbb{Z}^k$-subaction with positive entropy or a contradiction. We deduce that $\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) \geq \mathrm{kdim}(R_d/\mathfrak{p})$, and so $\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) = \mathrm{kdim}(R_d/\mathfrak{p})$.

Finally, suppose $\mathrm{char}(R_d/\mathfrak{p}) = p > 0$, and let $k = \mathrm{kdim}(R_d/\mathfrak{p})$. Then $R_d/\mathfrak{p}$ is a ring extension of $\mathbb{F}_p[u_1^{\pm 1}, \ldots, u_k^{\pm 1}]$. By Noether normalization [**S**, Section 8] we may choose variables so that $u_1, \ldots, u_k$ do not satisfy a polynomial relation, and the variables $u_{k+1}^{\pm 1}, \ldots, u_d^{\pm 1}$ are integral over $u_1, \ldots, u_k$. This shows that there is an expansive, positive-entropy $\mathbb{Z}^k$-subaction. Hence $\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) = \mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}) = \mathrm{kdim}(R_d/\mathfrak{p})$. □

The next example shows that the inequality in Proposition 7.3 can be strict when $\mathrm{char}(R_d/\mathfrak{p}) = 0$.

Example 7.4. (Krull dimension strictly less than expansive rank) Let $\phi(z) = z^2 - z - i = (z - \lambda_1)(z - \lambda_2)$ and $\psi(z) = z^2 + z - 2i = (z - \mu_1)(z - \mu_2)$, where $i = \sqrt{-1}$.

Set $F = \{0, \lambda_1, \lambda_2, \mu_1, \mu_2\}$. Define $\mathbf{s}\colon \mathbb{C} \to \mathbb{C}^3$ by $\mathbf{s}(z) = (z, \phi(z), \psi(z))$. Put

$$W = \{\mathbf{s}(z) : z \in \mathbb{C} \smallsetminus F\} \subset (\mathbb{C}^\times)^3.$$

Let $\mathfrak{p} \subset R_3$ be the ideal of Laurent polynomials in R_3 that vanish on all of W.

We will show that $\mathfrak{p}$ is a prime ideal, that $\alpha_{R_3/\mathfrak{p}}$ is expansive and mixing, and that $2 = \mathrm{kdim}(R_3/\mathfrak{p}) < \mathrm{exprk}(\alpha_{R_3/\mathfrak{p}}) = 3$.

To begin, let $g(u, v, w) = (u^2 - u - v)^2 + 1$ and $h(u, v, w) = w - 2v + u^2 - 3u$. We will show that $\mathfrak{p} = \langle g, h\rangle$. Both g and h vanish on W, and so $\langle g, h\rangle \subset \mathfrak{p}$. To establish the reverse inclusion, suppose that $f \in \mathfrak{p}$. We can reduce f in R_3 modulo $\langle h\rangle$ to have the form $w^{-n}k(u, v)$ for some $n \in \mathbb{Z}$ and $k \in R_2$. We can then reduce $k(u, v)$ modulo $\langle g\rangle$ in R_2 to have the form $v^{-m}[p(u)v + q(u)]$ for some $m \in \mathbb{Z}$ and $p(u), q(u) \in R_1$. Hence f is congruent in R_3 to $v^{-m}w^{-n}[p(u)v + q(u)]$ modulo $\langle g, h\rangle$. Since f is in $\mathfrak{p}$, it vanishes on W, so that

$$f(\mathbf{s}(z)) = \phi(z)^{-m}\psi(z)^{-n}[p(z)\phi(z) + q(z)] = 0 \quad \text{for } z \in \mathbb{C} \smallsetminus F.$$

Hence as Laurent polynomials with complex coefficients, $p(z)\phi(z) = -q(z)$. If $p \neq 0$ in R_1, then the non-zero coefficient of the smallest power of z on the left-hand side would be pure imaginary, while on the right-hand side it would be real. This contradiction shows that $p = 0$, and thus $q = 0$, hence $f \in \langle g, h\rangle$. This proves that $\mathfrak{p} = \langle g, h\rangle$.

To show that $\mathfrak{p}$ is prime, suppose that $f\cdot k \in \mathfrak{p}$ for some $f, k \in R_3$. Then $f(\mathbf{s}(z))\cdot k(\mathbf{s}(z))$ is a rational function that vanishes on $\mathbb{C} \smallsetminus F$. Hence at least one of $f(\mathbf{s}(z))$ or $k(\mathbf{s}(z))$ must vanish on $\mathbb{C} \smallsetminus F$, so that $f \in \mathfrak{p}$ or $k \in \mathfrak{p}$, establishing primality of $\mathfrak{p}$. Alternatively, one can give an algebraic proof that $\mathfrak{p}$ is prime along the lines of Example 5.8.

To compute $\mathrm{kdim}(R_3/\mathfrak{p})$, first note that h gives w in terms of u and v in $R_3/\mathfrak{p}$. Then g shows that v is algebraic over u in $R_3/\mathfrak{p}$. It now follows from standard facts about Krull dimension (see [**E**, Ch. 13]) that $\mathrm{kdim}(R_3/\mathfrak{p}) = \mathrm{kdim}(R_1) = 2$.

We next examine the expansive character of $\alpha_{R_3/\mathfrak{p}}$. Define $\overline{W} = \{(\overline{a}, \overline{b}, \overline{c}) : (a, b, c) \in W\}$. We claim that $\mathsf{V}(\mathfrak{p}) = W \cup \overline{W}$. Since all polynomials in $\mathfrak{p}$ have real coefficients, it is clear that $W \cup \overline{W} \subset \mathsf{V}(\mathfrak{p})$. Conversely, suppose that $(a, b, c) \in \mathsf{V}(\mathfrak{p})$. Since $g(a, b, c) = 0$, it follows that $a^2 - a - b = \pm i$. If $a^2 - a - b = i$, then using $h(a, b, c) = 0$ we see that $c = a^2 + a - 2i$. Thus $(a, b, c) = \mathbf{s}(a) \in W$. If $a^2 - a - b = -i$, then taking complex conjugates and applying the previous argument shows that $(\overline{a}, \overline{b}, \overline{c}) = \mathbf{s}(\overline{a}) \in W$, so that $(a, b, c) \in \overline{W}$. Hence $\mathsf{V}(\mathfrak{p}) = W \cup \overline{W}$. Note that $\log|W| = \log|\overline{W}|$, so that $\log|\mathsf{V}(\mathfrak{p})| = \log|W|$.

By Theorem 3.1, $\alpha_{R_3/\mathfrak{p}}$ is expansive if and only if $\mathbf{0} \notin \log|W|$. Hence it suffices to show that the curve $\gamma(\theta) = (\log|\phi(e^{i\theta})|, \log|\psi(e^{i\theta})|)$ does not pass through the origin. But this is clear from the graph of γ shown in Figure 6(a). Hence $\alpha_{R_3/\mathfrak{p}}$ is expansive.

Next, we show that $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}}) = \mathsf{S}_2$, and hence that $\mathsf{N}(\alpha_{R_3/\mathfrak{p}}) = \mathsf{S}_2$. Consider that cross-section of $\log|W|$ corresponding to setting the first coordinate to $\log r$. Since then $z = re^{i\theta}$, this cross-section is given by the curve $\gamma_r(\theta) = (\log|\phi(re^{i\theta})|, \log|\psi(re^{i\theta})|)$. For r close to 1 each γ_r is a curve that surrounds the origin, and hence the part of $\log|W|$ with first coordinate close to 0 is a surface that surrounds the origin in $\mathbb{R}^3$. As r varies from 0 to ∞, this surface generates six 'tendrils' stretching off to infinity (see Figure 6(b), where the term 'amoeba' is apt). Two tendrils correspond to $r = |\lambda_1| \approx 1.44$ and $r = |\lambda_2| \approx$

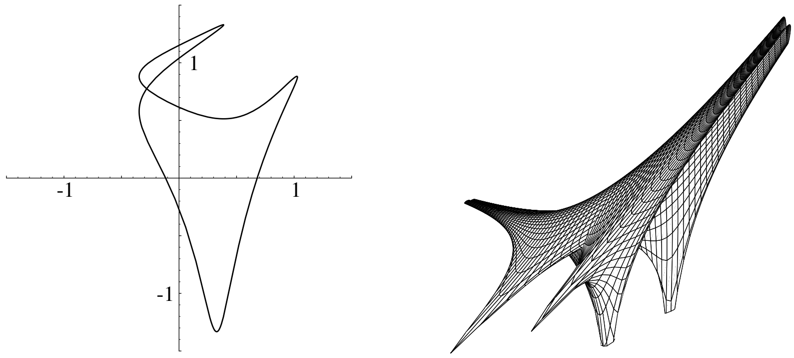


FIGURE 6. Cross-section and logarithmic image.

0.69, and they are asymptotic to the half-lines L_1 and L_2 in the $-\mathbf{e}_2$ direction given by $(\log|\lambda_j|, 0, \log|\psi(\lambda_j)|) - [0,\infty)\mathbf{e}_2$ for $j = 1, 2$. Another two tendrils correspond to the half-lines L_3 and L_4 in the $-\mathbf{e}_3$ direction given by $(\log|\mu_k|, \log|\phi(\mu_k)|, 0) - [0,\infty)\mathbf{e}_3$ for $k = 1, 2$. As $r \to 0$, $\log r \to -\infty$, and there is a fifth tendril asymptotic to the half-line L_5 in the $-\mathbf{e}_1$ direction given by $(0, \log|\phi(0)|, \log|\psi(0)|) - [0,\infty)\mathbf{e}_1$. Finally, as $r \to \infty$ there is a sixth tendril asymptotic to $L_6 = [0,\infty)\mathbf{v}$, where $\mathbf{v} = (1/3, 2/3, 2/3)$.

Note that when the half-lines L_1 through L_5 are extended to lines, none passes through the origin. It follows that the radial projection of $\log|W|$ to S_2 must cover all of S_2, with the possible exception of $\mathbf{v} = (1/3, 2/3, 2/3)$ corresponding to L_6. At this point we can conclude that $\mathsf{N}(\alpha_{R_3/\mathfrak{p}}) = \mathsf{S}_2$ since it is closed. However, we continue with the complete description of $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}})$.

To handle the remaining point $\mathbf{v}$, we show that there is a $z \in \mathbb{C} \smallsetminus F$ such that $\log|\phi(z)| = \log|\psi(z)| = 2\log|z|$. The equation $\log|\phi(z)| = 2\log|z| = \log|z^2|$ implies that $|\phi(z)/z^2| = 1$, so that $\phi(z)/z^2 = e^{i\theta}$ for some θ. Hence $(1 - e^{i\theta})z^2 - z - i = 0$, with roots

$$z = \frac{1 \pm \sqrt{1 + 4i(1 - e^{i\theta})}}{2(1 - e^{i\theta})}. \tag{7.1}$$

A similar calculation starting with $\log|\psi(z)| = 2\log|z|$ shows that

$$z = \frac{-1 \pm \sqrt{1 + 8i(1 - e^{i\xi})}}{2(1 - e^{i\xi})} \tag{7.2}$$

for some ξ. Plotting the roots using the positive sign for (7.1) and (7.2) gives the graphs shown in Figure 7. They cross in two points, one of which is $z_1 \approx 0.53 + 3.36i$. Then $z_1 \in \mathbb{C} \smallsetminus F$, $\log|z_1| > 0$, $\log|\phi(z_1)| > 0$, $\log|\psi(z_1)| > 0$, and so $\log|\mathbf{s}(z_1)| = t\mathbf{v}$ for some $t > 0$. Hence $\mathbf{v} \in \mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}})$, and so $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}}) = \mathsf{S}_2$. Thus $\operatorname{exprk}(\alpha_{R_3/\mathfrak{p}}) = 3$, showing that here Krull dimension is strictly less than expansive rank.

We conclude this example with two further properties of $\alpha_{R_3/\mathfrak{p}}$: it is mixing, and every 2-dimensional plane has positive entropy.

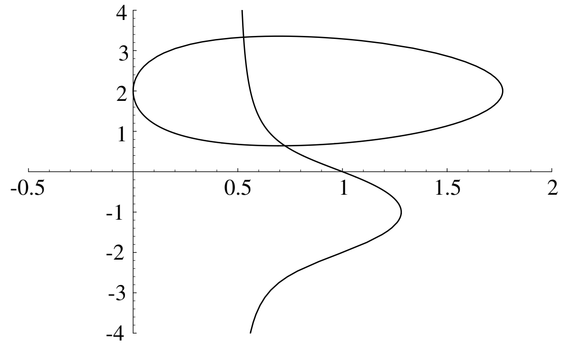


FIGURE 7. Graphs of roots.

By Theorem 6.5(2) of [**S**], $\alpha_{R_3/\mathfrak{p}}$ is mixing if and only if $u^{\mathbf{n}} - 1 \notin \mathfrak{p}$ for all $\mathbf{n} \in \mathbb{Z} \setminus \{\mathbf{0}\}$. Suppose the contrary. Then the plane $\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{n} = 0\}$ would contain $\log |W|$, which is evidently false. Thus $\alpha_{R_3/\mathfrak{p}}$ is mixing.

Let Q be a rational 2-dimensional subspace of $\mathbb{R}^3$. Denote the lattice $Q \cap \mathbb{Z}^3$ by Γ, and let $\mathbf{m}, \mathbf{n} \in \Gamma$ form an integral basis for Γ. Suppose that $\mathsf{h}(\alpha_{R_3/\mathfrak{p}}, \Gamma) = 0$. Let $R_\Gamma = \mathbb{Z}[u^{\mathbf{k}} : \mathbf{k} \in \Gamma]$ and $\mathfrak{p}_\Gamma = \mathfrak{p} \cap R_\Gamma$. As Γ-actions, $\alpha_{R_\Gamma/\mathfrak{p}_\Gamma}$ is a factor of $\alpha_{R_3/\mathfrak{p}}$, so that $\mathsf{h}(\alpha_{R_\Gamma/\mathfrak{p}_\Gamma}) = 0$. By the preceding paragraph, $\mathfrak{p}_\Gamma$ contains no generalized cyclotomic polynomials, so by Theorem 4.2 of [**LSW**], it follows that $\mathfrak{p}_\Gamma$ cannot be principal. Hence $\operatorname{kdim}(R_\Gamma/\mathfrak{p}_\Gamma) = 1$ and $R_\Gamma/\mathfrak{p}_\Gamma$ is certainly $\mathbb{Z}$-torsion-free, so that $\operatorname{kdim}[(R_\Gamma/\mathfrak{p}_\Gamma)\otimes\mathbb{Q}] = 0$. Thus $\mathsf{V}(\mathfrak{p}_\Gamma)$ is finite, say $\mathsf{V}(\mathfrak{p}_\Gamma) = \{(\xi_j, \eta_j) : 1 \le j \le r\}$. Hence for every $\mathbf{z} \in \mathsf{V}(\mathfrak{p})$ there is a $1 \le j \le r$ for which $\mathbf{z}^{\mathbf{m}} = \xi_j$ and $\mathbf{z}^{\mathbf{n}} = \eta_j$, or $\mathbf{m} \cdot \log |\mathbf{z}| = \log |\xi_j|$ and $\mathbf{n} \cdot \log |\mathbf{z}| = \log |\eta_j|$. It follows that if L_j denotes the line of intersection between the two planes $\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{m} \cdot \mathbf{x} = \log |\xi_j|\}$ and $\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{n} \cdot \mathbf{x} = \log |\eta_j|\}$, then $\log |\mathsf{V}(\mathfrak{p})| \subset \bigcup_{j=1}^r L_j$. But the radial projection of finitely many lines to S_2 cannot cover S_2, contradicting $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}}) = \mathsf{S}_2$. This shows that $\alpha_{R_3/\mathfrak{p}}$ has strictly positive entropy with respect to every rational 2-dimensional plane.

Example 7.4 shows that when $\operatorname{kdim}(R_d/\mathfrak{p}) = d-1$, then $\mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}})$ can have non-empty interior. We now turn to $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$. We have already observed two cases when $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ has non-empty interior: $\mathfrak{p} = 0$, with $\operatorname{kdim}(R_d/\mathfrak{p}) = d + 1$; and $\mathfrak{p} = \langle f \rangle$ with $|c_f(\mathbf{n})| > 1$ for some vertex $\mathbf{n}$ of $\mathcal{N}(f)$ (see Example 5.4), where $\operatorname{kdim}(R_d/\mathfrak{p}) = d$. Our next result shows that when $\operatorname{kdim}(R_d/\mathfrak{p})$ drops below d, then $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ cannot have interior.

PROPOSITION 7.5. *Suppose that* $\mathfrak{p}$ *is a prime ideal in* R_d *with* $\operatorname{kdim}(R_d/\mathfrak{p}) \le d-1$. *Then* $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ *is closed with empty interior.*

Proof. Remark 4.5 shows that $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ is closed.

First suppose that $\operatorname{char}(R_d/\mathfrak{p}) = p > 0$. Let $R_{d,p} = (\mathbb{Z}/p\mathbb{Z})[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$, and $\mathfrak{q}$ be the image of $\mathfrak{p}$ under the map $R_d \to R_{d,p}$ that reduces coefficients mod p. Then $R_d/\mathfrak{p} \cong R_{d,p}/\mathfrak{q}$. Hence $\operatorname{kdim}(R_{d,p}/\mathfrak{q}) = \operatorname{kdim}(R_d/\mathfrak{p}) \le d - 1 < d = \operatorname{kdim}(R_{d,p})$, so that $\mathfrak{q} \ne 0$. Choose $0 \ne f \in \mathfrak{q}$. Then $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ is contained in $\mathsf{N}^{\mathsf{n}}(\alpha_{R_{d,p}/\langle f \rangle})$, which by [**BL**, Theorem 7.2] equals the $(d-2)$-skeleton of the spherical dual of the mod p Newton polyhedron of f. Hence $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ has empty interior.

Finally, suppose that $\mathrm{char}(R_d/\mathfrak{p}) = 0$. Recall from Definition 6.4 the meaning of support and of H-exposed vertex.

Let $\mathcal{U}$ be an arbitrary non-empty open set in H_d. Choose $a \geq 1$ minimal so that there is an $H \in \mathcal{U}$ and $f \in \mathfrak{p}$ having an H-exposed vertex $\mathbf{n}$ with $c_f(\mathbf{n}) = a$. Note that $\mathfrak{p} \neq 0$ since $\mathrm{kdim}(R_d/\mathfrak{p}) \leq d-1$, so that a exists. We claim that $a = 1$. If so, then Lemma 4.4 shows that $\mathcal{U}$ contains an H for which $R_d/\mathfrak{p}$ is R_H-Noetherian, and so $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ is nowhere dense.

Suppose that $a > 1$. Choose $H \in \mathcal{U}$ and $f \in \mathfrak{p}$ such that f has an H-exposed vertex $\mathbf{n}$ with $c_f(\mathbf{n}) = a$. We may clearly assume that f is irreducible since any factor of f would also have an H-exposed vertex. Since H-exposure is an open condition and rational directions are dense, we may also assume that H is rational, i.e. that there is an $\mathbf{m} \in \mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$ with $H = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{m} \leq 0\}$. Adjusting f by a monomial if necessary, we may assume that $\mathrm{supp}(f) \subset H$ and that $\mathrm{supp}(f) \cap \partial H = \{\mathbf{0}\}$, so that $c_f(\mathbf{0}) = a$.

Next, we claim that there is a $0 \neq g \in \mathfrak{p}$ with $\mathrm{supp}(g) \subset \partial H$. Equivalently, if $R_{\mathbf{m}}$ denotes $\mathbb{Z}[u^{\mathbf{k}} : \mathbf{k} \cdot \mathbf{m} = 0]$, then $R_{\mathbf{m}} \cap \mathfrak{p} \neq \{0\}$. For suppose that $R_{\mathbf{m}} \cap \mathfrak{p} = \{0\}$. Since f is irreducible and $R_d/\langle f \rangle$ has Krull dimension d, it follows that $0 \subsetneqq \langle f \rangle \subsetneqq \mathfrak{p}$ is a chain of prime ideals in R_d. Let S denote the multiplicatively closed subset $R_{\mathbf{m}} \smallsetminus \{0\}$ of R_d. Since $S \cap \mathfrak{p} = \varnothing$, it follows that $0 \subsetneqq S^{-1}(\langle f \rangle) \subsetneqq S^{-1}\mathfrak{p}$ is a chain of prime ideals in $S^{-1}R_d$. But $S^{-1}R_d \cong S^{-1}R_{\mathbf{m}}[v, v^{-1}]$ for a suitable monomial v, and $S^{-1}R_{\mathbf{m}}$ is a field, so that $\mathrm{kdim}(S^{-1}R_{\mathbf{m}}[v, v^{-1}]) = 1$, contradicting the existence of a chain of primes of length two. Hence $R_{\mathbf{m}} \cap \mathfrak{p} \neq \{0\}$.

Let $0 \neq g \in R_{\mathbf{m}} \cap \mathfrak{p}$. Since $\mathrm{char}(R_d/\mathfrak{p}) = 0$, then $\mathfrak{p} \cap \mathbb{Z} = \{0\}$. If all the coefficients of g are divisible by a, then primality of $\mathfrak{p}$ shows that g/a is also in $\mathfrak{p}$. Hence we can find $g \in R_{\mathbf{m}} \cap \mathfrak{p}$ not all of whose coefficients are divisible by a. Let

$$h = g - \sum_{\mathbf{n} \in \mathrm{supp}(g)} \left\lfloor \frac{c_g(\mathbf{n})}{a} \right\rfloor u^{\mathbf{n}} f.$$

Since $\mathrm{supp}(g) \subset \partial H$, $\mathrm{supp}(f) \cap \partial H = \{\mathbf{0}\}$, and $c_f(\mathbf{0}) = a$, it follows that $h \in \mathfrak{p}$, $\mathrm{supp}(h) \subset H$, $\mathrm{supp}(h) \cap \partial H \neq \varnothing$, and $0 < c_h(\mathbf{n}) < a$ for every $\mathbf{n} \in \mathrm{supp}(h) \cap \partial H$. Hence there is a small perturbation $H' \in \mathcal{U}$ of H for which h has an H'-exposed vertex $\mathbf{n}$ with $0 < c_h(\mathbf{n}) < a$. This contradicts the minimality of a, proving that $a = 1$, and completing the proof. □

8. *Lower-dimensional subspaces*

Thus far we have concentrated on expansive behavior along half-spaces and their $(d-1)$-dimensional boundaries. Once this is found, then expansive behavior along lower-dimensional subspaces is completely determined by [**BL**, Theorem 3.6]: a k-plane is non-expansive for a $\mathbb{Z}^d$-action if and only if it is contained in a non-expansive $(d-1)$-plane (or, equivalently, a non-expansive half-space). Additionally, half-spaces H give rise to subrings R_H of R_d, and this algebraic structure makes certain arguments work smoothly.

In this section we sketch how to modify our definitions and proofs to work for lower-dimensional subspaces. In particular, we obtain a direct description of the set $\mathsf{N}_k(\alpha_M)$

of non-expansive k-planes as the union of two pieces, one coming from a Noetherian condition on M along k-planes, and the other from a variety condition involving the orthogonal complements of k-planes. This description is a lower-dimensional version of (6.4).

We begin by defining a notion of Noetherian along a general subspace of $\mathbb{R}^d$.

Definition 8.1. Let M be a Noetherian R_d-module and V be a k-plane in $\mathbb{R}^d$. Then M is *Noetherian along* V if M is R_H-Noetherian for every half-space $H \in \mathsf{H}_d$ containing V. The set of k-planes V along which M is not Noetherian is denoted by $\mathsf{N}_k^{\mathfrak{n}}(\alpha_M)$.

Remark 8.2. By Remark 4.5, $\{H \in \mathsf{H}_d : M \text{ is } R_H\text{-Noetherian}\}$ is open. It follows that $\{V \in \mathsf{G}_k : M \text{ is Noetherian along } V\}$ is open, so that $\mathsf{N}_k^{\mathfrak{n}}(\alpha_M)$ is closed in G_k.

In the following, recall that V^t denotes the thickening of a subspace V by an amount t.

LEMMA 8.3. *Let M be a Noetherian R_d-module and V be a subspace of $\mathbb{R}^d$. Then M is Noetherian along V if and only if there are $m_1, \dots, m_r \in M$ and $t > 0$ such that M is generated as a group by $\{u^{\mathbf{n}} m_j : 1 \le j \le r \text{ and } \mathbf{n} \in V^t \cap \mathbb{Z}^d\}$. In particular, if V is rational, then M is Noetherian along V if and only if M is Noetherian as a module over the ring $\mathbb{Z}[u^{\mathbf{n}} : \mathbf{n} \in V \cap \mathbb{Z}^d\}$.*

Proof. The case $V = 0$ is the implication (1a) $\Rightarrow$ (1b) in Proposition 6.9. The case of general V uses an entirely analogous adaptation of the proof of [**BL**, Theorem 3.6].

If V is rational, let $R_V = \mathbb{Z}[u^{\mathbf{n}} : \mathbf{n} \in V \cap \mathbb{Z}^d]$. Then R_V is a Noetherian ring. If M is generated as a group by $\{u^{\mathbf{n}} m_j : 1 \le j \le r, \mathbf{n} \in V^t \cap \mathbb{Z}^d\}$, then it is a finitely-generated R_V-module, hence Noetherian over R_V. Conversely, if M is Noetherian over R_V and $H \in \mathsf{H}_d$ contains V, then trivially M is R_H-Noetherian as well, so that M is Noetherian along V. □

For a subspace V of $\mathbb{R}^d$, let $V^\perp$ denote its orthogonal complement.

THEOREM 8.4. *Let M be a Noetherian R_d-module, α_M be the corresponding algebraic $\mathbb{Z}^d$-action, and V be a subspace of $\mathbb{R}^d$. Then the following are equivalent:*
(1) *α_M is expansive along V;*
(2) *$\alpha_{R_d/\mathfrak{p}}$ is expansive along V for every $\mathfrak{p} \in \operatorname{asc}(M)$;*
(3) *$R_d/\mathfrak{p}$ is Noetherian along V and $V^\perp \cap \log|\mathsf{V}(\mathfrak{p})| = \varnothing$ for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

Proof. (1)$\Leftrightarrow$(2): The case $V = \mathbb{R}^d$ is exactly Theorem 3.1. We may therefore assume that $\dim V \le d-1$. By [**BL**, Theorem 3.6], α_M is not expansive along V if and only if there is a $W \in \mathsf{G}_{d-1}$ containing V such that α_M is not expansive along W. By Lemma 2.9, this occurs if and only if there is an $H \in \mathsf{N}(\alpha_M)$ containing V. Theorem 4.9 shows that this happens if and only if $H \in \mathsf{N}(\alpha_{R_d/\mathfrak{p}})$ for some $\mathfrak{p} \in \operatorname{asc}(M)$. Reversing the chain of equivalences, this time for $\alpha_{R_d/\mathfrak{p}}$, shows that this occurs if and only if $\alpha_{R_d/\mathfrak{p}}$ is not expansive along V for some $\mathfrak{p} \in \operatorname{asc}(M)$.

(2)$\Leftrightarrow$(3): As in the proof of (1)$\Leftrightarrow$(2), $\alpha_{R_d/\mathfrak{p}}$ is not expansive along V if and only if there is an $H \in \mathsf{N}(\alpha_{R_d/\mathfrak{p}})$ containing V. Theorem 4.9 shows that this occurs if and only if either $R_d/\mathfrak{p}$ is not R_H-Noetherian or $[0,\infty)\mathbf{v}_H \cap \log|\mathsf{V}(\mathfrak{p})| \ne \varnothing$, where $\mathbf{v}_H$ is the outward unit normal for H. By definition, $\alpha_{R_d/\mathfrak{p}}$ is not Noetherian along V if and only if there is a half-space H containing V for which $R_d/\mathfrak{p}$ is not R_H-Noetherian. If $V \subset H$, then $\mathbf{v}_H \in V^\perp$. Hence there is an $H \in \mathsf{H}_d$ containing V with $[0,\infty)\mathbf{v}_H \cap \log|\mathsf{V}(\mathfrak{p})| \neq \varnothing$ if and only if $V^\perp \cap \log|\mathsf{V}(p)| \neq \varnothing$. □

If M is a Noetherian R_d-module and V is a k-plane, then Remark 6.2 and Definition 8.1 show that M is Noetherian along V if and only if $R_d/\mathfrak{p}$ is Noetherian along V for every $\mathfrak{p} \in \mathrm{asc}(M)$. Hence

$$\mathsf{N}_k^{\mathsf{n}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\mathrm{asc}(M)} \mathsf{N}_k^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}}). \tag{8.1}$$

Let us define

$$\mathsf{N}_k^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}) = \{V \in \mathsf{G}_k : V^\perp \cap \log|\mathsf{V}(\mathfrak{p})| \neq \varnothing\}$$

and

$$\mathsf{N}_k^{\mathsf{v}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\mathrm{asc}(M)} \mathsf{N}_k^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}).$$

Then Theorem 8.4 says that

$$\mathsf{N}_k(\alpha_M) = \mathsf{N}_k^{\mathsf{n}}(\alpha_M) \cup \mathsf{N}_k^{\mathsf{v}}(\alpha_M).$$

When $k = d-1$, this is the image of the equality (6.4) under the map $\pi : \mathsf{H}_d \to \mathsf{G}_{d-1}$ defined by $\pi(H) = \partial H$.

Next, we prove a lower-dimensional version of Proposition 7.5.

PROPOSITION 8.5. *Let M be a Noetherian R_d-module, and let $k \leq d$ be such that* $\mathrm{kdim}(R_d/\mathfrak{p}) \leq k$ *for every* $\mathfrak{p} \in \mathrm{asc}(\alpha_M)$. *Then* $\mathsf{N}_k^{\mathsf{n}}(\alpha_M)$ *is a closed subset of* G_k *with empty interior.*

Proof. $\mathsf{N}_k^{\mathsf{n}}(\alpha_M)$ is closed by Remark 8.2. The case $d = k$ is trivial. Suppose that $k = d-1$. Let $\pi : \mathsf{H}_d \to \mathsf{G}_{d-1}$ be $\pi(H) = \partial H$. By definition, $\pi(\mathsf{N}^{\mathsf{n}}(\alpha_M)) = \mathsf{N}_{d-1}^{\mathsf{n}}(\alpha_M)$. Since $\mathsf{N}^{\mathsf{n}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\mathrm{asc}(M)} \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ and each $\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}})$ is nowhere dense in H_d by Proposition 7.5, it follows that $\mathsf{N}_{d-1}^{\mathsf{n}}(\alpha_M)$ is nowhere dense.

We prove the general case by downward induction on k. Assume the result is established for $k+1$. Let $\mathcal{V}$ be an open set in G_k. Then there is an open set $\mathcal{W}$ in G_{k+1} such that every $W \in \mathcal{W}$ contains a $V \in \mathcal{V}$. By the induction hypothesis, $\mathcal{W}$ contains a rational $(k+1)$-plane W along which M is Noetherian. By Lemma 8.3, M is Noetherian over $R_W = \mathbb{Z}[u^{\mathbf{n}} : \mathbf{n} \in W \cap \mathbb{Z}^d]$. We are now in the codimension one situation of the first part of the proof. The associated primes of M as an R_W-module are $\mathfrak{p} \cap R_W$ for $\mathfrak{p} \in \mathrm{asc}(M)$. Then $R_W/(\mathfrak{p} \cap R_W)$ is a subring of $R_d/\mathfrak{p}$, hence the hypothesis on Krull dimension is satisfied for this situation. Thus there is a $V \in \mathcal{V}$ along which M is Noetherian. This proves that $\mathsf{N}_k^{\mathsf{n}}(\alpha_M)$ is nowhere dense in G_k. □

Example 8.6. We can now give another way to determine the non-Noetherian set N^{n} in Example 5.8 that does not use the 2-adic arguments there. The polynomial $w - 2 \in \mathfrak{p}$ shows that N^{n} is contained in the lower hemisphere, and then $1 + u + v \in \mathfrak{p}$ shows that N^{n} is contained in the union Q of the three quarter-meridians shown in Figure 5(b).

Assume that for some $\mathbf{v}$ in Q the module $R_3/\mathfrak{p}$ is Noetherian over $R_{H_\mathbf{v}}$. Since the Noetherian set is open, we may assume that $\mathbf{v}$ is rational and does not lie on the equator. Then there is a great circle C through $\mathbf{v}$ which does not contain any other point of Q. We may assume that there is an $\mathbf{n} \in \mathbb{Z}^d$ such that C is the intersection of S_2 and the plane orthogonal to $\mathbf{n}$.

By Definition 8.1, the module $R_3/\mathfrak{p}$ is Noetherian along $V = \mathbb{R}\mathbf{n}$, since every half-space $H_\mathbf{w}$ containing V has $\mathbf{w} \in C$. By Lemma 8.3, $R_3/\mathfrak{p}$ is Noetherian over the ring $S = \mathbb{Z}[u^{\pm\mathbf{n}}]$.

If $S \cap \mathfrak{p} \neq \{0\}$, then $\mathrm{kdim}(S/(\mathfrak{p} \cap S)) = 1$, and since $R_3/\mathfrak{p}$ is an integral extension we would have $\mathrm{kdim}(R_3/\mathfrak{p}) = 1$, a contradiction. If $S \cap \mathfrak{p} = \{0\}$, then $w^{-1} = 1/2 \in R_3/\mathfrak{p}$ cannot be integral over S, contradicting the fact that $R_3/\mathfrak{p}$ is Noetherian over S. Hence $\mathsf{N}^{\mathfrak{n}}$ consists of all of Q.

9. *Homoclinic points and groups*

For the first part of this section we return to topological $\mathbb{Z}^d$-actions. Recall from §2 that V^t denotes the thickening of a subspace V by t.

Definition 9.1. Let β be a $\mathbb{Z}^d$-action on a compact metric space (X, ρ). Suppose that V is a subspace of $\mathbb{R}^d$ and that $y_0 \in X$. We say that $x \in X$ is *homoclinic to y_0 along V* if there is a $t > 0$ such that $\rho(\alpha^\mathbf{n}(x), \alpha^\mathbf{n}(y_0)) \to 0$ as $\|\mathbf{n}\| \to \infty$ and $\mathbf{n} \in V^t$. The set of points in X homoclinic to y_0 along V is denoted by $\Delta_\beta(y_0, V)$. In case $V = \mathbb{R}^d$ we delete the phrase 'along V', and write $\Delta_\beta(y_0)$ for $\Delta_\beta(y_0, \mathbb{R}^d)$.

Remark 9.2. If there is some $t_0 > 0$ for which $\rho(\alpha^\mathbf{n}(x), \alpha^\mathbf{n}(y_0)) \to 0$ as $\|\mathbf{n}\| \to \infty$ and $\mathbf{n} \in V^{t_0}$, then this also holds along V^t for every $t > 0$. This follows easily from the observation that $\mathbb{Z}^d \cap V^{t_0}$ has bounded gaps together with continuity of β.

PROPOSITION 9.3. *Let V be an expansive subspace for the topological $\mathbb{Z}^d$-action β on X, and $y_0 \in X$. Then $\Delta_\beta(y_0, V)$ is at most a countable set.*

Proof. This can be proven exactly as in [**LS**, Lemma 3.2]. □

Example 9.4. Let $\mathcal{A}$ be a finite alphabet, $X = \mathcal{A}^{\mathbb{Z}^d}$, and β be the $\mathbb{Z}^d$-shift action on X. Then β is expansive, and $\Delta_\beta(y_0)$ is the countable set of points in X differing from y_0 in only finitely many coordinates. For every subspace V of dimension $< d$, it is easy to see that $\Delta_\beta(y_0, V)$ is uncountable.

Example 9.5. We return to Ledrappier's example (see Examples 2.5 and 5.6). Here $\beta = \alpha_{R_2/\langle 2,1+u+v\rangle}$ and $X = X_{R_2/\langle 2,1+u+v\rangle}$. We choose $y_0 = 0_X = 0$. Since the only point in X having finitely many non-zero coordinates is 0, we see that $\Delta_\beta(0) = \{0\}$. Let L_θ denote the line in $\mathbb{R}^2$ making angle θ with the positive horizontal axis. For all θ with $0 < \theta < \pi/2$, the sets $\Delta_\beta(0, L_\theta)$ are all equal. Each consists of points of the form indicated in Figure 8(a), where the shaded regions contain only 0's, and coordinates represented by the dots can be of an arbitrary finite length, filled in with arbitrary values, and these are used to determine all remaining coordinates.

For $\pi/2 < \theta < 3\pi/4$, all the sets $\Delta_\beta(0, L_\theta)$ are also equal, and each consists of points shown in Figure 8(b), with the same conventions as before. For $3\pi/4 < \theta < \pi$,

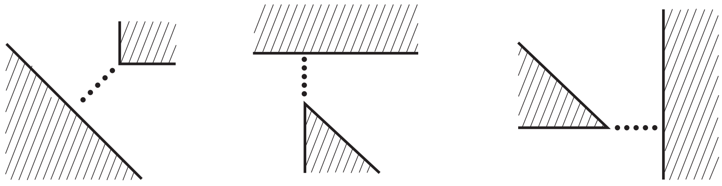


FIGURE 8. Homoclinic points for Ledrappier's example.

Figure 8(c) describes the form of points in each $\Delta_\beta(0, L_\theta)$. It is also easy to see that $\Delta_\beta(0, L_0) = \Delta_\beta(0, L_{\pi/2}) = \Delta_\beta(0, L_{3\pi/4}) = \{0\}$.

Here β has expansive components $\mathcal{C}_1 = \{L_\theta : 0 < \theta < \pi/2\}$, $\mathcal{C}_2 = \{L_\theta : \pi/2 < \theta < 3\pi/4\}$, and $\mathcal{C}_3 = \{L_\theta : 3\pi/4 < \theta < \pi\}$. Within an expansive component the homoclinic set is constant, but it changes abruptly when passing from one component to another. See [**MS**] for another example of how the homoclinic group varies.

The following result shows that homoclinic sets are always constant within an expansive component. Roughly speaking, nearby planes in an expansive component code each other, so a point that is homoclinic for one is also homoclinic for the other.

THEOREM 9.6. *Let β be a topological $\mathbb{Z}^d$-action on a compact metric space X, let $\mathcal{C}$ be a connected component of $\mathsf{E}_k(\beta)$, and let $y_0 \in X$. Then $\Delta_\beta(y_0, V) = \Delta_\beta(y_0, W)$ for all $V, W \in \mathcal{C}$.*

Proof. We show that $\Delta_\beta(y_0, \cdot)$ is locally constant on $\mathcal{C}$, hence constant on $\mathcal{C}$ by connectedness.

Let $U \in \mathcal{C}$ and δ be an expansive constant for β. Recall Definition 2.8 of coding for subsets of $\mathbb{R}^d$. Let $B(r)$ be the open ball of radius r in $\mathbb{R}^d$. For $V \in \mathsf{G}_k$ and $\eta > 0$ put $V_\eta = \{\mathbf{x} \in \mathbb{R}^d : \operatorname{dist}(\mathbf{x}, V) < \eta\|\mathbf{x}\|\}$. It follows from [**BL**, Proposition 3.8] that there is a compact neighborhood $\mathcal{U}$ of U in $\mathcal{C}$ and positive numbers $s, t, \eta > 0$ such that for every $V, W \in \mathcal{U}$ and every $r > 0$ we have that $V^t \setminus B(r)$ codes $W_\eta \setminus B(r + s)$.

Suppose that $V, W \in \mathcal{U}$ and $x \in \Delta_\beta(y_0, V)$. Then there is an $r > 0$ such that $\rho(\beta^{\mathbf{n}}(x), \beta^{\mathbf{n}}(y_0)) \le \delta$ for every $\mathbf{n} \in V^t \setminus B(r)$. Hence by coding, this inequality also holds for every $\mathbf{n} \in W_\eta \setminus B(r + s)$. Let $\varepsilon > 0$. A simple compactness argument using expansiveness of β shows that there is an $a > 0$ such that $\rho_\beta^{B(a)}(x, y_0) \le \delta$ implies that $\rho(x, y_0) < \varepsilon$. Choose q large enough so that if $\mathbf{x} \in W$ and $\|\mathbf{x}\| > q$, then $\mathbf{x} + B(a + t) \subset W_\eta \setminus B(r + s)$. It follows that $\rho(\beta^{\mathbf{n}}(x), \beta^{\mathbf{n}}(y_0)) < \varepsilon$ for every $\mathbf{n} \in W^t$ with $\|\mathbf{n}\| > q$. This proves that $x \in \Delta_\beta(y_0, W)$, and so $\Delta_\beta(y_0, V) \subset \Delta_\beta(y_0, W)$. Interchanging the roles of V and W gives the reverse inclusion. □

We next turn to considering homoclinic points for an algebraic $\mathbb{Z}^d$-action α on X. Since ρ is translation invariant, $\rho(\alpha^{\mathbf{n}}(x), \alpha^{\mathbf{n}}(y_0)) = \rho(\alpha^{\mathbf{n}}(x - y_0), 0_X)$, so that $\Delta_\alpha(y_0, V) = y_0 + \Delta_\alpha(0_X, V)$. Hence for algebraic $\mathbb{Z}^d$-actions we need only compute $\Delta_\alpha(0_X, V)$, which we shorten to $\Delta_\alpha(V)$. Obviously $\Delta_\alpha(V)$ is a subgroup of X, which we term the *homoclinic group of α along V*. When $V = \mathbb{R}^d$, the subgroup $\Delta_\alpha = \Delta_\alpha(\mathbb{R}^d)$ is called the *homoclinic group of α*.

The homoclinic groups of algebraic $\mathbb{Z}^d$-actions were studied in [**LS**], especially for expansive actions. The main result there is the following [**LS**, Theorems 4.1 and 4.2]. Here 'entropy' is taken with respect to Haar measure.

THEOREM 9.7. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on X.*
(1) *Δ_α is non-trivial if and only if α has positive entropy.*
(2) *Δ_α is non-trivial and dense in X if and only if α has completely positive entropy.*

Using this together with constancy of the homoclinic group within an expansive component, we obtain further instances of the expansive subdynamics philosophy, as follows.

Call $V \in \mathsf{G}_k$ *rational* if $V \cap \mathbb{Z}^d$ spans V. For an expansive component $\mathcal{C} \subset \mathsf{E}_k(\alpha)$ let $\mathcal{C}_\mathbb{Q} = \{V \in \mathcal{C} :$ V is rational$\}$, which is a dense subset of $\mathcal{C}$. For $V \in \mathcal{C}_\mathbb{Q}$ we let $\alpha|_{V\cap\mathbb{Z}^d}$ denote the rank k action obtained by restricting α to $V \cap \mathbb{Z}^d$.

THEOREM 9.8. *Let α be an algebraic $\mathbb{Z}^d$-action and $\mathcal{C}$ be an expansive component of $\mathsf{E}_k(\alpha)$. If for some $V \in \mathcal{C}_\mathbb{Q}$ the action $\alpha|_{V\cap\mathbb{Z}^d}$ has positive entropy, has completely positive entropy, or is isomorphic to a $\mathbb{Z}^k$ Bernoulli shift, then the same property holds for $\alpha|_{W\cap\mathbb{Z}^d}$ for every $W \in \mathcal{C}_\mathbb{Q}$.*

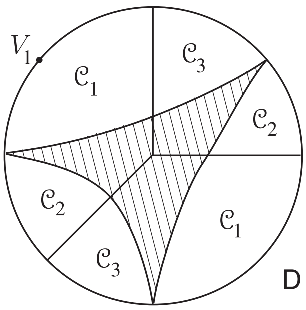
Proof. By Theorem 9.7, $\alpha|_{V\cap\mathbb{Z}^d}$ has positive entropy if and only if $\Delta_\alpha(V) \neq \{0_X\}$, and by Theorem 9.6 we have $\Delta_\alpha(V) = \Delta_\alpha(W)$ for all $W \in \mathcal{C}_\mathbb{Q}$, establishing the positive entropy portion. The proof for completely positive entropy is similar, using density rather than non-triviality of the homoclinic group. Finally, completely positive entropy for an algebraic $\mathbb{Z}^d$-action is equivalent to Bernoullicity (see [**RS**] or [**S**, §23]). □

Remarks 9.9. (1) By [**BL**, Theorem 6.3(4)], if there is a rational $V \in \mathsf{E}_k(\alpha)$ for which $\alpha|_{V\cap\mathbb{Z}^d}$ has zero entropy, then $\alpha|_{W\cap\mathbb{Z}^d}$ has zero entropy for every $W \in \mathsf{G}_k$. This not only provides an alternative proof of the first part of the previous theorem, it also shows that on the entire set $\mathsf{E}_k(\alpha)$ either entropy vanishes identically or it is strictly positive everywhere.

(2) It is likely that suitable definitions of completely positive entropy and Bernoullicity can be found for non-rational k-planes, and that the conclusions of Theorem 9.8 can be extended to all $W \in \mathcal{C}$. However, we will not pursue this further here.

Example 9.10. (Homoclinic groups of 2-planes for Example 5.8) To illustrate some of the phenomena surrounding the preceding theorems, we describe the homoclinic groups of 2-planes for Example 5.8. In this example, $d = 3$, $\mathfrak{p} = \langle 1+u+v, w-2\rangle$, $X = X_{R_3/\mathfrak{p}}$, and $\alpha = \alpha_{R_3/\mathfrak{p}}$. Observe that since $\mathfrak{p}$ is non-principal, α has zero entropy. Since α is expansive, $\Delta_\alpha = \{0\}$ by Theorem 9.7.

To help describe the groups $\Delta_\alpha(V)$ for 2-planes $V \in \mathsf{G}_2$, recall the map $\pi : \mathsf{H}_3 \to \mathsf{G}_2$ defined by $\pi(H) = \partial H$. Under our standing correspondence $\mathsf{H}_3 \leftrightarrow \mathsf{S}_2$, this map is the usual identification of antipodal points of S_2 to obtain projective 2-space G_2. By vertically projecting the upper hemisphere of S_2 to the unit disk D, we can represent G_2 as D with antipodal boundary points identified. Using this representation, the image $\pi(\mathsf{N}(\alpha))$ in D is shown in Figure 9(a). The shaded region is $\pi(\mathsf{N}^{\mathsf{v}}(\alpha))$, while the three segments comprise $\pi(\mathsf{N}^{\mathsf{n}}(\alpha))$. There are three expansive components of 2-planes, labeled $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$.



(a)

$$
\begin{array}{cccccccccc}
 & & & & \vdots & & & & & \\
 & -c & -3c & -3c & -c & 0 & 0 & 0 & 0 & \\
 & 0 & c & 2c & c & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & -c & -c & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & -c & c & -c & c & \\
 & 0 & 0 & 0 & 0 & 0 & c & -2c & 3c & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 3c & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & \\
 & & & & \vdots & & & & &
\end{array}
$$

(b)

FIGURE 9. Analysis for homoclinic groups of 2-planes.

We say that $y \in \Delta_\alpha(V)$ is a *fundamental homoclinic point for* V if $\{\alpha^{\mathbf{n}}(y) : \mathbf{n} \in \mathbb{Z}^3\}$ generates $\Delta_\alpha(V)$ as an abelian group.

We will sketch the following description of how $\Delta_\alpha(V)$ varies for $V \in \mathsf{D}$.

(1) For each expansive component $\mathcal{C}_r$ there is an explicit non-trivial fundamental homoclinic point x^r (recall that $\Delta_\alpha(\cdot)$ is constant within an expansive component);

(2) $\Delta_\alpha(V) = \{0\}$ for every $V \in \pi(\mathsf{N}^{\mathsf{n}}(\alpha))$; and

(3) for every $V \in \pi(\mathsf{N}^{\mathsf{v}}(\alpha)) \setminus \pi(\mathsf{N}^{\mathsf{n}}(\alpha))$ there is a non-trivial fundamental homoclinic point x^V that varies continuously with V.

For (1), first consider component $\mathcal{C}_1$. Define $x^1 \in X$ by

$$
x^1_{i,j,k} = \begin{cases} (-1)^{i-1}\dbinom{i}{-j}2^k, & \text{if } i \geq 0,\ j \leq 0, \\[2ex] (-1)^{j-1}\dbinom{j-1}{-i-1}2^k, & \text{if } i \leq -1,\ j \geq 1, \\[2ex] 0, & \text{otherwise.} \end{cases}
$$

Figure 9(b) shows a typical horizontal slice of x^1 at level k, where $c = 2^k$. The 2-plane $V_1 \in \mathcal{C}_1$, shown in Figure 9(a), is the vertical plane that intersects the horizontal uv-plane in the line $u = v$. Clearly $x^1 \in \Delta_\alpha(V_1)$. Using the fact that the coordinates of any point in X homoclinic along a vertical line must be an integer times the successive powers of 2, it is not hard to show that integer combinations of shifts of x^1 comprise all of $\Delta_\alpha(V_1)$. Thus x^1 is a fundamental homoclinic point for V_1. It is therefore also a fundamental homoclinic point for all $V \in \mathcal{C}_1$. A similar construction produces fundamental homoclinic points x^2 and x^3 for $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively, which bear the same relationship to x^1 as Figures 8(b) and 8(c) bear to Figure 8(a).

It is also possible to show using this description that the intersection of the homoclinic groups of any pair of distinct expansive components is trivial, analogous to a result in [**MS**] for a pair of commuting toral automorphisms.

For (2), we give a complete argument only for the horizontal plane V_0. A somewhat more complicated proof is required for general $V \in \pi(\mathsf{N}^{\mathsf{n}}(\alpha))$. For brevity we assume familiarity with [**LS**].

Suppose that $x \in \Delta_\alpha(V_0)$. Let $f(u,v) = 1 + u + v$, and define $\widetilde{f} \in \ell^\infty(\mathbb{Z}^3, \mathbb{Z})$ by $\widetilde{f}_{\mathbf{n}} = c_f(-\mathbf{n})$. Choose $\overline{x} \in \ell^\infty(\mathbb{Z}^3, \mathbb{R})$ so that $|\overline{x}_{\mathbf{n}}| \leq \frac{1}{2}$ and $\overline{x}_{\mathbf{n}} \equiv x_{\mathbf{n}} \pmod 1$. Then $\overline{x}_{\mathbf{n}} \to 0$ along each horizontal plane $V_k = V_0 + k\mathbf{e}_3$. Since $x \in X$, we have that $\widetilde{f} * \overline{x} \in \ell^\infty(\mathbb{Z}^3, \mathbb{Z})$ and also tends to 0 in each V_k. Hence there are polynomials $h_k \in R_2/\langle 1+u+v\rangle$ such that $\widetilde{f} * \overline{x}|_{V_k} = (w^k h_k)^{\sim}$. Since $w - 2 \in \mathfrak{p}$, it follows that $2h_{k-1} \equiv h_k$ in $R_2/\langle 1+u+v\rangle$. Hence each h_k is divisible in $R_2/\langle 1+u+v\rangle$ by arbitrarily large powers of 2. But $R_2/\langle 1+u+v\rangle \cong \mathbb{Z}[u^{\pm 1}, 1/(u+1)]$ is a localization of $\mathbb{Z}[u]$ in which 2 is not invertible. Hence each $h_k = 0$. Let $\overline{y} \in \ell^\infty(\mathbb{Z}^2, \mathbb{R})$ be the restriction of $\overline{x}$ to some V_k. Then $\widetilde{f} * \overline{y} = 0$, so that $\overline{y}$ is a pseudo-measure on $\mathbb{Z}^2$ whose support must be contained in the finite set $\mathsf{V}(f) \cap \mathbb{S}^2$. But then $\overline{y}$ is almost periodic, and in particular $\overline{y} \notin c_0(\mathbb{Z}^2)$ unless $\overline{y} = 0$. This proves that the only point homoclinic for V_0 is $x = 0$.

For (3), let $V \in \pi(\mathsf{N}^{\mathsf{v}}(\alpha)) \smallsetminus \pi(\mathsf{N}^{\mathsf{n}}(\alpha))$. Choose a, b so that $(a, b, 1)$ is normal to V. Put $f_{a,b}(u,v) = 1 + 2^a u^{-1} + 2^b v^{-1}$ and $F_{a,b} = 1/f_{a,b}$. To construct x^V, we first construct an auxiliary point $y \in \ell^\infty(\mathbb{Z}^2, \mathbb{R})$ as follows. If $V \notin \partial\pi(\mathsf{N}^{\mathsf{v}}(\alpha))$ then $F_{a,b}$ has two poles of order one in $\mathbb{S}^2$, and so $F_{a,b} \in L^1(\mathbb{S}^2)$. In this case we put $y_{\mathbf{n}} = \widehat{F_{a,b}}(\mathbf{n})$. Then

$$y_{\mathbf{n}} \to 0 \quad \text{as} \quad \|\mathbf{n}\| \to \infty \tag{9.1}$$

by the Riemann–Lebesgue lemma. Also,

$$(\widetilde{f}_{a,b} * y)_{\mathbf{n}} = y_{\mathbf{n}} + 2^a y_{\mathbf{n}+\mathbf{e}_1} + 2^b y_{\mathbf{n}+\mathbf{e}_2} = \begin{cases} 1, & \text{if } \mathbf{n} = \mathbf{0}, \\ 0, & \text{otherwise.} \end{cases} \tag{9.2}$$

If $V \in \partial\pi(\mathsf{N}^{\mathsf{v}}(\alpha))$, we can still expand $F_{a,b}$ in a series. For example, if $a < 0$ and $b < 0$, then V being on the boundary of $\pi(\mathsf{N}^{\mathsf{v}}(\alpha))$ corresponds to $2^a + 2^b = 1$. Hence

$$\begin{aligned} F_{a,b} = \frac{1}{1 + 2^a u^{-1} + 2^b v^{-1}} &= \sum_{k=0}^{\infty} (-1)^k (2^a u^{-1} + 2^b v^{-1})^k \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^k \binom{k}{j} 2^{aj + b(k-j)} u^{-j} v^{-(k-j)}. \end{aligned}$$

In this case let $y_{(m,n)}$ denote the coefficient of $u^m v^n$ in the expansion. Then (9.1) holds by standard estimates and (9.2) by construction.

Next we construct x^V from y. Let $V' = V - [0,1)\mathbf{e}_3$. Vertical projection from $\mathbb{R}^3$ to $\mathbb{R}^2$ gives a bijection from $V' \cap \mathbb{Z}^3$ to $\mathbb{Z}^2$. Let $\{r\}$ denote the fractional part of a real number r. Define $x^V_{\mathbf{n}}$ for $\mathbf{n} \in V' \cap \mathbb{Z}^3$ by

$$x^V_{(m,n,\lfloor -ma-nb \rfloor)} = 2^{-\{-ma-nb\}} y_{(m,n)}.$$

Extend x^V to those $\mathbf{n}$ above V' using $x^V_{\mathbf{n}+\mathbf{e}_3} = 2x^V_{\mathbf{n}}$, and to those $\mathbf{n}$ below V' using $x^V_{\mathbf{n}} + x^V_{\mathbf{n}+\mathbf{e}_1} + x^V_{\mathbf{n}+\mathbf{e}_2} = 0$. The former extension is clearly unique, and the latter is unique since $V \notin \pi(\mathsf{N}^{\mathsf{n}}(\alpha))$. An elementary argument shows that $x^V \in X$, and x^V is homoclinic along V by construction. It is not difficult to establish that x^V is a fundamental homoclinic point for V. Finally, the construction shows that x^V varies continuously for $V \in \pi(\mathsf{N}^{\mathsf{v}}(\alpha)) \smallsetminus \pi(\mathsf{N}^{\mathsf{n}}(\alpha))$.

Acknowledgements. The first author was supported in part by FWF Research Grant P12250–MAT and EPSRC Grant GR/M 49588. The second author was supported in part by NSF Grant DMS-9622866. The first two authors gratefully acknowledge support from the Schrödinger Institute at the University of Vienna. The third author gratefully acknowledges support from EPSRC Grant 97700813. The fourth author thanks the University of East Anglia and the American Mathematical Society for their support.

REFERENCES

[AL] W. W. Adams and P. Loustaunau. *An Introduction to Gröbner Bases*. American Mathematical Society, Providence, RI, 1994.

[B] G. M. Bergman. The logarithmic limit-set of an algebraic variety. *Trans. Amer. Math. Soc.* **157** (1971), 459–469.

[BG] R. Bieri and J. R. J. Groves. The geometry of the set of characters induced by valuations. *J. Reine Angew. Math.* **347** (1984), 168–195.

[BL] M. Boyle and D. Lind. Expansive subdynamics. *Trans. Amer. Math. Soc.* **349** (1997), 55–102.

[EW] M. Einsiedler and T. Ward. Fitting ideals for finitely presented algebraic dynamical systems. *Aequationes Math.* **60** (2000), 57–71.

[E] D. Eisenbud. *Commutative Algebra with a View toward Algebraic Geometry*. Springer, New York, 1995.

[GKZ] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants*. Birkhäuser, Boston, 1994.

[L] S. Lang. *Algebra*, 2nd edn. Addison-Wesley, Menlo Park, 1984.

[LS] D. Lind and K. Schmidt. Homoclinic points of algebraic $\mathbb{Z}^d$-actions. *J. Amer. Math. Soc.* **12** (1999), 953–980.

[LSW] D. Lind, K. Schmidt and T. Ward. Mahler measure and entropy for commuting automorphisms of compact groups. *Invent. Math.* **101** (1990), 593–629.

[MS] A. Manning and K. Schmidt, Common homoclinic points of commuting toral automorphisms. *Israel J. Math* **114** (1999), 289–300.

[M] R. Miles. Arithmetic dynamical systems. *PhD Thesis*, University of East Anglia, 2000.

[RS] D. J. Rudolph and K. Schmidt. Almost block independence and Bernoullicity of $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups. *Invent. Math.* **120** (1995), 455–488.

[S] K. Schmidt. *Dynamical Systems of Algebraic Origin*. Birkhäuser, 1995.

[Sc] E. Scott. Expansive subdynamics of algebraic actions generated by principal ideals in the ring of two-variable Laurent polynomials. *Master's Thesis*, University of Washington, 1999.

[Sh] M. Shereshevsky. Expansiveness, entropy and polynomials growth for groups acting on subshifts by automorphisms. *Indag. Math.* **4**(2) (1993), 203–210.

[St] B. Sturmfels. *Gröbner Bases and Convex Polytopes*. American Mathematical Society, Providence, RI, 1996.