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ENTROPIES OF AUTOMORPHISMS OF A TOPOLOGICAL MARKOV SHIFT

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ABSTRACT. Let σ be a mixing topological Markov shift, λ a weak Perron number, $q(t)$ a polynomial with nonnegative integer coefficients, and r a nonnegative rational. We construct a homeomorphism commuting with σ whose topological entropy is $\log[q(\lambda)q(1/\lambda)]^r$. These values are shown to include the logarithms of all weak Perron numbers, and are dense in the nonnegative reals.

1. Introduction. Let A be a square nonnegative integral matrix. We assume throughout that A is aperiodic in that some power of A is strictly positive. A familiar procedure [**Wi**] associates to A a homeomorphism σ_A of a totally disconnected compact space X_A. Aperiodicity of A means that σ_A is topologically mixing. Such homeomorphisms are called topological Markov shifts or subshifts of finite type. For a discussion of their central role in dynamics, see [**DGS**].

Denote by $\operatorname{aut}(\sigma_A)$ the group of homeomorphisms of X_A commuting with σ_A. The case $A=[n]$ was studied by Hedlund and coworkers [**H**], whose principal motivation appears to have been cryptographic. The algebraic complexity of $\operatorname{aut}(\sigma_{[n]})$ is indicated by Hedlund's theorem that it contains an isomorphic copy of every finite subgroup. The proof easily extends to general σ_A. Although $\operatorname{aut}(\sigma_A)$ has been recently studied by several authors [**Wa**, **BK**, **BLR**], its algebraic structure remains mostly a mystery. No one knows, for example, whether $\operatorname{aut}(\sigma_{[2]})$ is generated by $\sigma_{[2]}$ and the involutions in the group.

By the Curtis-Lyndon-Hedlund theorem [**H**], each $\varphi \in \operatorname{aut}(\sigma_A)$ is induced by a block map. Thus the topological entropy $h(\varphi)$ is finite. Such an automorphism φ can be regarded as a bijective cellular automaton map [**Wo**], and $h(\varphi)$ measures the information flow of the map. What are the possible values of $h(\varphi)$ for $\varphi \in \operatorname{aut}(\sigma_A)$? Denote this set by H_A. Despite the finite nature of shift commuting maps, there are essentially no nontrivial exact computations of their entropy in the literature. One exception is Coven's calculation of the entropy of a map commuting with the one-sided 2-shift [**C**]. In particular, it was unknown whether or not H_A is dense in $[0,\infty)$.

Our purpose here is to construct automorphisms of σ_A with interesting entropies. We will obtain values, like $\log(9/2)$, that are sometimes not logarithms of algebraic integers, and which form a dense subset of $[0,\infty)$.

To state the precise results, call λ a Perron number (and write $\lambda \in \mathbf{P}$) if $\lambda \geq 1$ is an algebraic integer that strictly exceeds the absolute value of its other conjugates.

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Call $\lambda \geq 1$ weak Perron ($\lambda \in W\mathbf{P}$) if some nonnegative integral power of it is Perron. Let $\mathbf{Q}^+ = \{r \in \mathbf{Q}: r \geq 0\}$ and $\mathbf{Z}^+ = \mathbf{Z} \cap \mathbf{Q}^+$. In [**L**] it is shown that $\mathbf{P}$ coincides with the set of spectral radii of aperiodic $\mathbf{Z}^+$-matrices, and $W\mathbf{P}$ is the set of spectral radii of arbitrary $\mathbf{Z}^+$-matrices. Let

$$\Lambda = \{[q(\lambda)q(\lambda^{-1})]^r : \lambda \in W\mathbf{P},\ q(t) \in \mathbf{Z}^+[t],\ r \in \mathbf{Q}^+\}.$$

In Theorem 1 we show that if $\alpha \in \Lambda$, there is a $\varphi \in \operatorname{aut}(\sigma_A)$ with $h(\varphi) = \log \alpha$. In Theorem 2 we prove that $W\mathbf{P} \subset \Lambda$. Thus our results are summarized by

$$\log W\mathbf{P} \subset \log \Lambda \subset H_A$$

for every aperiodic A.

The construction uses markers to define two involutions in $\operatorname{aut}(\sigma_A)$. Their composition moves buffers that reserve differing lengths of space for storing information. This movement causes background data from an auxiliary topological Markov shift to be shifted by various amounts. The whole marker-buffer-data system is topologically conjugate to a certain skew product whose entropy can be calculated using results of Marcus and Newhouse [**MN**], leading to the numbers above. One consequence is the existence of elements in $\operatorname{aut}(\sigma_A)$ with exactly two measures of maximal entropy. This construction is also used in [**BLR**] to embed the countable direct sum of infinite cyclic groups into the automorphism group of every mixing topological Markov shift.

Most of the ideas used in our construction are contained in §2. There we find, for fixed $q(t) \in \mathbf{Z}^+[t]$, an automorphism of a special full shift whose entropy is $\log[q(2)q(\frac{1}{2})]$. In §3 we show how to adapt and elaborate these ideas to prove Theorem 1. Some elementary estimates in §4 prove Theorem 2.

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2. Constructing automorphisms. Suppose $q(t) \in \mathbf{Z}^+[t]$. We construct an automorphism φ of the full $q(1)+3$ shift with $h(\varphi) = \log[q(2)q(\frac{1}{2})]$.

If $q(t) = q_d t^d + \cdots + q_1 t + q_0$ with $q_i \in \mathbf{Z}^+$, let

$$L = \{0, 1, m\} \cup \{b_{ij} : 0 \leq i \leq d,\ 1 \leq j \leq q_i\}.$$

If $q_i = 0$, there are no symbols b_{ij}. Let σ_L denote the full L-shift on X_L. Using juxtaposition to denote concatenation of symbols, create buffers of length $i+1$ from each b_{ij} by putting $B_{ij} = b_{ij}^{i+1}$. We use $i+1$ rather than i to allow several buffers when $i = 0$.

Next we define two involutions φ_1 and φ_2 in $\operatorname{aut}(\sigma_L)$. Consider blocks of the form $mB_{ij}DB_{rs}m$ with data $D \in \{0,1\}^{2d-i-r+3}$. For such blocks $|D| \geq 3$, where $|D|$ denotes the length of the string D. We declare that φ_1 replaces $mB_{ij}DB_{rs}m$ with $mB_{rs}DB_{ij}m$ and vice versa, has no other effect. Clearly $\varphi_1^2 = I$, the identity, so $\varphi_1 \in \operatorname{aut}(\sigma_L)$. The definition of φ_2 is more complicated. Let a *frame* be a block of the form

$$mB_{i_0j_0}D_0B_{r_0s_0}m^*B_{i_1j_1}D_1B_{r_1s_1}m,$$

where $D_k \in \{0,1\}^{2d-i_k-r_k+3}$ $(k = 0,1)$, and * denotes the central symbol. We declare φ_2 to be a block map acting only on frames, and taking the above frame to $D_{10}B_{i_1j_1}m^*B_{r_0s_0}$ if $r_0 \geq i_1$, where $D_1 = D_{10}D_{11}$ with $D_{10} \in \{0,1\}^{r_0-i_1}$, and to

$B_{i_1j_1}m^*B_{r_0s_0}D_{01}$ if $r_0 \le i_1$, where $D_0 = D_{00}D_{01}$ with $D_{01} \in \{0,1\}^{i_1-r_0}$. Since φ_2 does not disturb markers and affects only symbols within $d+1$ of m^*, while markers in a frame are spaced $2d+3$ apart, it follows that φ_2 is well defined. Although φ_2 can affect the entries in a frame, it does not affect the property of being a frame. Hence $\varphi_2^2 = I$, and $\varphi_2 \in \operatorname{aut}(\sigma_L)$.

Let $\varphi = \varphi_2\varphi_1$. To compute $h(\varphi)$, we first construct a φ-invariant compact set Y such that $h(\varphi) = h(\varphi|_Y)$. We then observe that $\varphi|_Y$ is topologically conjugate to a certain skew product, whose entropy we compute using [**MN**].

Let

$$Y = \{x \in X_L \colon x = (\cdots mB_{i_{-1}j_{-1}}D_{-1}B_{r_{-1}s_{-1}}m^*B_{i_0j_0}D_0B_{r_0s_0}m\cdots)\},$$

where * denotes the 0th coordinate, and

$$D_n \in \{0,1\}^{2d-i_n-r_n+3}, \qquad n \in \mathbf{Z}.$$

Then φ acts on Y as follows. The buffer $B_{i_nj_n}$ is shifted right to replace $B_{i_{n+1}j_{n+1}}$, and $B_{r_ns_n}$ is shifted left to replace $B_{r_{n-1}s_{n-1}}$. The background data in D_0 is shifted by $i_0 - r_0$ to the left by φ_1, and not shifted by φ_2. Thus near the 0th coordinate, the background data in the infinite string $\cdots D_{-1}D_0D_1\cdots$ maintains its order and is shifted by $i_{-n+1} - r_{n-1}$ under φ^n.

If the defining pattern for Y is broken, buffers shifting towards the break are reflected to return in the opposite direction. Background data near such a break can move by at most a bounded amount for all iterates of φ.

To initiate the computation of $h(\varphi)$, let $\mathcal{A}$ partition X_L by the 0th coordinate. For $p \ge 0$ let

$$\mathcal{A}^p = \bigvee_{|j|\le p} \sigma_L^j(\mathcal{A}),$$

where empty intersections are discarded. Then

$$h(\varphi, \mathcal{A}^p) = \lim_{N\to\infty} \frac{1}{N} \log \left| \bigvee_{j=0}^{N} \varphi^{-j}(\mathcal{A}^p) \right|,$$

and since $\mathcal{A}$ generates under σ_L,

$$h(\varphi) = \lim_{p\to\infty} h(\varphi, \mathcal{A}^p).$$

We assume $p > 3d+6$, and estimate the cardinality of $\bigvee_{j=0}^N \varphi^{-j}(\mathcal{A}^p)$ as follows. For $x \in X_L$ define $x[a,b]$ to be the block $x_ax_{a+1}\cdots x_b$. Say a block B occurs in a subshift $Z \subset X_L$ (and write $B \in Z$) if there is a $z \in Z$ and a, b so that $z[a,b] = B$. First suppose F is a nonempty atom of $\bigvee_{j=0}^N \varphi^{-j}(\mathcal{A}^p)$, and that for $x \in F$ we have $x[-p,p] \notin Y_0 = \bigcup_{k=0}^{2d+5} \sigma_L^k Y$. By the description of the action of φ given above, we see that $x[-p,p]$ contains a break acting as a reflecting barrier to buffers and data. It follows that $(\varphi^jx)[-p,p]$ can only contain new information from a possible new buffer entering from the right, another entering from the left, and a bounded amount of data. Since there are $q(1)$ different buffers, we obtain

$$\left| \bigvee_{j=0}^{N} \varphi^{-j}(\mathcal{A}^p) \cap Y_0^c \right| = O(q(1)^{2N}).$$

Thus $h(\varphi|_{Y_0^c}) \leq \log q(1)^2$. Now $h(\varphi|_{Y_0}) = \max\{h(\varphi|_{\sigma_L^k Y})\colon 0 \leq k \leq 2d+5\} = h(\varphi|_Y)$, and we will show that $h(\varphi|_Y) = \log[q(2)q(\frac{1}{2})]$. Hence the evaluation of $h(\varphi)$ follows from the next result.

LEMMA. *Let $q(t) \in \mathbf{Z}^+[t]$. Then*

$$\frac{d^2}{dt^2}\left[q(t)q\left(\frac{1}{t}\right)\right] \in \mathbf{Z}^+[t, t^{-1}].$$

Hence $q(t)q(1/t)$ is increasing and convex on $[1, \infty)$.

PROOF. Put $f(t) = q(t)q(1/t) = \sum_{j=-d}^{d} c_j t^j$, where $c_j \in \mathbf{Z}^+$. Then

$$f''(t) = \sum_{j=-d}^{d} j(j-1)c_j t^{j-2} \in \mathbf{Z}^+[t, t^{-1}].$$

Since $f'(1) = 0$, the second statement follows. □

To compute $h(\varphi|_Y)$, consider $B = \{B_{ij}\colon 0 \leq i \leq d,\ 1 \leq j \leq q_i\}$ as a set of abstract symbols. Let σ_B denote the shift on the space X_B of all sequences of the B_{ij}. For $u = (B_{i_k j_k}\colon k \in \mathbf{Z}) \in X_B$ and $v = (B_{r_k s_k}\colon k \in \mathbf{Z})$, define $g\colon X_B \times X_B \to \mathbf{Z}$ by $g(u,v) = i_0 - r_0$. Define the skew product $\tau = (\sigma_B \times \sigma_B^{-1}) \times \sigma_2^g$ on $Z = X_B \times X_B \times X_2$ by

$$\tau(u, v, w) = (\sigma_B u, \sigma_B^{-1} v, \sigma_2^{g(u,v)} w).$$

Our next goal is to prove $\varphi|_Y$ is conjugate to τ.

Define $\theta\colon Y \to Z$ by

$$\begin{aligned}\theta(\cdots m^* B_{i_0 j_0} D_0 B_{r_0 s_0} m \cdots) \\ = ((\cdots B^*_{i_0 j_0} B_{i_1 j_1} \cdots), (\cdots B^*_{r_0 s_0} B_{r_1 s_1} \cdots), (\cdots D_0^* D_1 \cdots)),\end{aligned}$$

where D_0^* indicates the 0th coordinate at symbol $d+1-i_0$ of D_0. Note that on the left side $B_{i_k j_k}$ denotes a block of symbols from L, while on the right it is considered as a single symbol from the alphabet B. Clearly θ is a continuous bijection from Y to Z, and from our description of $\varphi|_Y$ above it follows that $\theta\varphi = \tau\theta$, so θ conjugates $\varphi|_Y$ with τ.

To complete an analysis of φ, we use [**MN**] to compute $h(\tau) = h(\varphi|_Y)$. By Theorem B of [**MN**],

$$h(\tau) = \max\{P(\sigma_B \times \sigma_B^{-1}, (\log 2)g), P(\sigma_B \times \sigma_B^{-1}, -(\log 2)g)\},$$

where P denotes topological pressure. These pressures are explicitly computable, coincide, and by [**MN**, p. 111] equal

$$\begin{aligned}&\log\left(\sum_{(B_{ij}, B_{rs}) \in B \times B} 2^{g(B_{ij}, B_{rs})}\right) \\ &= \log\left(\sum_{(B_{ij}, B_{rs})} 2^{i-r}\right) = \log\left[\left(\sum_{B_{ij}} 2^i\right)\left(\sum_{B_{rs}} 2^{-r}\right)\right] \\ &= \log\left[\left(\sum_{r=0}^{d} q_i 2^i\right)\left(\sum_{r=0}^{d} q_r 2^{-r}\right)\right] = \log\left[q(2)q\left(\frac{1}{2}\right)\right].\end{aligned}$$

This proves $h(\varphi) = \log[q(2)q(\frac{1}{2})]$.

We conclude this section by indicating a modification to obtain an automorphism φ with $h(\varphi) = r\log[q(2)q(\frac{1}{2})]$ for $r \in \mathbf{Q}^+$. By using powers, it suffices to find, for given $n \geq 1$, an automorphism with entropy $(1/n)\log[q(2)q(\frac{1}{2})]$. Fixing n, replace the symbol m with n distinct symbols $m_0, m_1, \ldots, m_{n-1}$, expanding L to

$$\{m_0, m_1, \ldots, m_{n-1}, 0, 1\} \cup \{b_{ij} : 0 \leq i \leq d,\ 1 \leq j \leq q_i\}.$$

Define φ_1 and φ_2 in $\operatorname{aut}(\sigma_L)$ as above with m replaced by m_0. Define φ_3 to be the 1-block map $\varphi_3(m_j) = m_{j+1(\operatorname{mod} n)}$. Put $\varphi = \varphi_3\varphi_2\varphi_1$. Let Y be the compact set above, except with m replaced by m_0. Then $\varphi^n|_Y = \varphi_2\varphi_1|_Y$, and estimates as before show $h(\varphi^n|_Y) = h(\varphi^n)$. Hence

$$nh(\varphi) = h(\varphi_2\varphi_1) = \log[q(2)q(1/2)],$$

concluding the verification that φ has the stated entropy.

3. Extension to general topological Markov shifts. The constructions in §2 took place on a special full shift. We show here how to imitate these ideas to construct automorphisms of a general mixing subshift of finite type, and how to employ background data from an arbitrary subshift of finite type instead of the 2-shift. This yields the automorphisms described in §1.

THEOREM 1. *Let σ_A be a mixing topological Markov shift, $\lambda \in W\mathbf{P}$, $q(t) \in \mathbf{Z}^+[t]$, and $r \in \mathbf{Q}^+$. There is an automorphism φ of σ_A with*

$$h(\varphi) = r\log[q(\lambda)q(1/\lambda)].$$

PROOF. We will construct $\varphi \in \operatorname{aut}(\sigma_A)$ with $h(\varphi) = \log[q(\lambda)q(1/\lambda)]$. By cycling markers and taking powers as in §2, it is possible to obtain this entropy multiplied by an arbitrary $r \in \mathbf{Q}^+$. The argument is the same, and will be omitted.

Since $\lambda \in W\mathbf{P}$, by [**L**] there is an irreducible 0-1 matrix E with spectral radius λ. Let $e = \dim E$, and set $N = q(1) + e + 1$. We first find N blocks of equal length in X_A so that no (not necessarily distinct) pair can overlap at all, with the trivial exception that each block entirely overlaps itself.

Since σ_A is mixing and nontrivial, there must be a loop $i_0 i_1 \cdots i_k i_0 \in X_A$ of distinct symbols with $k \geq 1$. Furthermore, one of these symbols, which we can assume is i_0, is followed by a symbol $j_1 \neq i_1$. First suppose $j_1 \neq i_0$. Choose a path of minimal length from j_1 to the loop, say $j_1 j_2 \cdots j_r j_s$. The case $r = 0$ is possible and corresponds to $j_1 = i_s$ for some $s \neq 0, 1$. Define $\alpha = i_0 \cdots i_k$, $\beta = i_0 j_1 j_2 \cdots j_r i_s \cdots i_k$. Then the N blocks $\alpha^2\beta^{r+1}\alpha\beta^{N+1-r}$ for $1 \leq r \leq N$ have no nontrivial overlaps. If $j_1 = i_0$, put $\alpha = i_1 \cdots i_r i_0$, and use $\alpha^2 i_0^r \alpha i_0^{N-r+1}$ for $1 \leq r \leq N$.

Label these blocks by μ, β_{ij} $(0 \leq i \leq d,\ 1 \leq j \leq q_i)$, and δ_i $(0 \leq i \leq e-1)$, and put $\Delta = \{\delta_i : 0 \leq i \leq e-1\}$. Then every point in X_A decomposes uniquely into nonoverlapping occurrences of these blocks, separated by other strings. Now define $\varphi = \varphi_2\varphi_1$ as follows. Put $B_{ij} = \beta_{ij}^{i+1}$. The automorphism φ_1 replaces $\mu B_{ij} D B_{rs} \mu$ with $\mu B_{rs} D B_{ij} \mu$ and vice versa, where $D = \delta_{i_1} \cdots \delta_{i_k}$, $k = 2d - i - r + 3$, and $i_1 \cdots i_k \in X_E$. To construct φ_2, define a *frame* to be a block of the form $\mu B_{i_0 j_0} D_0 B_{r_0 s_0} \mu^* B_{i_1 j_1} D_1 B_{r_1 s_1} \mu$, where $D_k \in \Delta^{2d - i_k - r_k + 3}$ $(k = 0, 1)$ and the subscripts of the δ's appearing in $D_0 D_1$ form a block in X_E. As in §2, φ_2

is defined to be a block map acting only on frames, and taking the above frame to $D_{10}B_{i_1j_1}\mu^*B_{r_0s_0}$ if $r_0 \geq i_1$, where $D_1 = D_{10}D_{11}$ with $D_{10} \in \Delta^{r_0-i_1}$, and to $B_{i_1j_1}\mu^*B_{r_0s_0}D_{01}$ if $r_0 \leq i_1$, where $D_0 = D_{00}D_{01}$ with $D_{01} \in \Delta^{i_1-r_0}$. Noting that such a block map preserves the frame property, including the condition that $D_0D_1 \in X_E$, the reasoning of §2 applies to prove that Δ_2 is a well-defined involution in $\operatorname{aut}(\sigma_A)$.

As in §2, set

$$Y = \{x \in X_A : x = (\cdots \mu^* B_{i_0j_0}D_0B_{r_0s_0}\mu\cdots)\},$$

where $|B_{i_kj_k}D_kB_{r_ks_k}| = (2d+5)|\mu|$ for $n \in \mathbf{Z}$ and $\cdots D_{-1}D_0D_1\cdots \in X_E$. Arguing as before, Y is φ-invariant, $h(\varphi) = h(\varphi|_Y)$, and $\varphi|_Y$ is conjugate to the skew product $\tau = (\sigma_B \times \sigma_B) \times \sigma_E^g$. The calculation of $h(\tau)$ in [**MN**] proves that

$$h(\varphi) = h(\varphi|_Y) = h(\tau) = P(\sigma_B \times \sigma_B^{-1}, (\log\lambda)g) = \log[q(\lambda)q(1/\lambda)],$$

completing the proof. □

4. Automorphisms with Perron entropies. The set Λ defined in §1 is a collection of algebraic numbers dense in $[1,\infty)$. Which algebraic integers are in Λ? The following is a partial answer.

THEOREM 2. *$W\mathbf{P} \subset \Lambda$.*

PROOF. The rough ideas are that for large $|t|$ the function $(1+t)(1+t^{-1})$ is essentially $t+2$, and that if $\lambda \in \mathbf{P}$ has the absolute value of its other conjugates $< \lambda - 4$, then $\lambda - 2 \in \mathbf{P}$.

Specifically, let $\pi \in W\mathbf{P}$ and choose k so that $\lambda = \pi^k \in \mathbf{P}$. Since Λ is closed under taking roots, we need only find n so that $\lambda^n \in \Lambda$. Let λ have conjugates $\lambda_1 = \lambda, \lambda_2, \ldots, \lambda_d$, where $|\lambda_j| < \lambda$ $(j \geq 2)$. Choose n large enough so $\lambda^n > |\lambda_j^n| + 6$ $(j \geq 2)$. The roots of

$$(*) \qquad (1+t)(1+t^{-1}) = z$$

are

$$f_\pm(z) = \tfrac{1}{2}\left(z - 2 \pm \sqrt{z^2 - 4z}\right),$$

where $\sqrt{z}$ denotes the standard branch. We will show that $\alpha = f_+(\lambda^n)$ is Perron, which will prove that $\pi = [(1+\alpha)(1+\alpha^{-1})]^{1/nk} \in \Lambda$.

Since λ is integral, $(*)$ with $z = \lambda^n$ is monic in t with integral coefficients, so α is integral. Note that

$$|f_\pm(z)| \leq \tfrac{1}{2}\left\{|z| + 2 + \sqrt{|z|^2 + 4|z|}\right\} \leq |z| + 2.$$

Furthermore, since $4/(\lambda^n - 2)^2 < 1$,

$$\alpha = f_+(\lambda^n) = \frac{1}{2}(\lambda^n - 2)\left\{1 + \sqrt{1 - \frac{4}{(\lambda^2 - 2)^2}}\right\}$$
$$\geq \frac{1}{2}(\lambda^n - 2)\left\{2 - \frac{4}{(\lambda^n - 2)^2}\right\} > \lambda^n - 3.$$

Now the other conjugates of α are either $f_-(\lambda^n)$ or have the form $f_\pm(\lambda_j^n)$ $(j \geq 2)$. Clearly $|f_-(\lambda^n)| < \alpha$, and since $|\lambda_j^n| + 6 < \lambda^n$ we have

$$|f_\pm(\lambda_j^n)| \leq |\lambda_j^n| + 2 < \lambda^n - 4 < f_+(\lambda^n) = \alpha,$$

proving $\alpha \in \mathbf{P}$ and completing the proof. □

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