Vol. 42, No. 4 DUKE MATHEMATICAL JOURNAL© December 1975

A COUNTEREXAMPLE TO A CONJECTURE OF HOPF

D. A. LIND

1. Introduction.

In one of a series of classical papers on ergodic theory, Hopf [4] discusses various mixing conditions that a measure-preserving flow $\{T_t\}$ on a Lebesgue measure space (X, μ) may obey. We list these in increasing order of strength. The flow is *ergodic* (Hopf's "metrically transitive") if whenever a measurable set A is invariant under every transformation T_t, then $\mu(A) = 0$ or $\mu(X \backslash A) = 0$; it is *totally ergodic* ("completely transitive") if every T_t with $t \neq 0$ is ergodic; it is *weakly mixing* if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\mu([T_tA] \cap B) - \mu(A)\mu(B)|\, dt = 0$$

for all measurable sets A and B; it is *mixing* if

$$\lim_{t\to\infty} \mu([T_tA] \cap B) = \mu(A)\mu(B)$$

for all A and B.

In [4] Hopf proves using spectral theory that a flow is totally ergodic if and only if it is weakly mixing, and that these conditions are equivalent to the flow having no eigenfunction with nonzero eigenvalue. Note that the analogous statement for transformations is false; a rotation of the unit circle by an irrational multiple of 2π has every nonzero power ergodic, but it is not weakly mixing. Hopf also says that he was not able to prove that a totally ergodic flow is mixing, although he has "little doubt about its being true." We construct here a counterexample.

A weakly mixing transformation that is not mixing was first constructed by Kakutani and von Neumann (unpublished). Chacon [2] gave a different and more geometric construction, and using ideas suggested by it showed that the speed of any ergodic flow could be altered to yield a weakly mixing flow. We show here how to make a continuous version of Chacon's construction.

2. The construction.

Our flow will be constructed by a continuous analogue of the cutting and stacking constructions of transformations (see [3; Chapter 6]). Let X_0 denote the unit square $[0, 1) \times [0, 1)$ equipped with planar Lebesgue measure μ. The flow $\{T_t\}$ is partially defined on X_0 by flowing a point at unit speed vertically until it reaches the top; symbolically, $T_t(x, y) = (x, y + t)$ for $-y \leq t < 1 - y$.

Received May 12, 1975.

To define $\{T_t\}$ on more of the space, we chop X_0 vertically into halves, place the right half on top of the left half, and add on top a piece of thickness d_1 to form a new space $X_1 = [0, \frac{1}{2}) \times [0, 2 + d_1)$. We regard X_0 as a subset of X_1 via this process, and observe that the flow partially defined on X_1 extends the previous one on X_0. In general, we obtain from X_n a larger space X_{n+1} by chopping X_n in two, placing the right half on top of the left half, and adding a piece of thickness d_{n+1} to it. Thus $X_n = [0, 2^{-n}) \times [0, h_n)$, where

$$h_n = 2^n + \sum_{i=1}^{n} 2^{n-i} d_i .$$

The amount added at the n^{th} stage has measure $2^{-n} d_n$. Regarding X_n as a subset of X_{n+1}, the direct limit space X of $\{X_n\}$ therefore has measure $1 + \sum_{n=1}^{\infty} 2^{-n} d_n$, which we will assume finite. Normalize the limit measure μ so that $\mu(X) = 1$. The flow $\{T_t\}$ is defined for all real t and all points of X except for the forward orbit of $(0, 0)$, a set of measure zero which we remove from X.

The flow that we have just constructed can easily be realized as a flow build under a function (for this fact in general, see [1]). The base transformation S on $[0, 1)$ is the "adding machine" described as follows. Let $x \in [0, 1)$ have the binary expansion $x = .\, a_1 a_2 \cdots = \sum a_i 2^{-i}$, $a_i = 0$ or 1, with terminating expansion if there is a choice. Let

$$S(.111 \cdots 10 a_{n+1} a_{n+2} \cdots) = (\,000 \cdots 01 a_{n+1} a_{n+2} \cdots).$$

Then S is an ergodic measure-preserving transformation on $[0, 1)$, which is in fact isomorphic to a translation on a compact abelian group. Let the function f on $[0, 1)$ be defined by

$$f(.111 \cdots 10 a_{n+2} a_{n+3} \cdots) = 1 + \sum_{i=1}^{n} d_i$$

Then the flow on $\{(x, y) : 0 \leq x < 1, 0 \leq y < f(x)\}$ defined by moving a point (x, y) vertically at unit speed until $(x, f(x))$, and then jumping to $(Sx, 0)$, is isomorphic to our flow $\{T_t\}$. From this, and the fact that S has zero entropy, it follows that $\{T_t\}$ also has zero entropy.

3. The flow is weakly mixing but not mixing.

We will show that no matter which d_n are chosen, $\{T_t\}$ is not mixing. However, if $\{d_n\}$ is dense in $(0, \infty)$, then the flow will be weakly mixing, and therefore also totally ergodic by Hopf's result. Sequences $\{d_n\}$ that are dense in $(0, \infty)$ and such that $\sum 2^{-n} d_n < \infty$ are easy to construct.

We will prove weak mixing by showing that the flow has no eigenfunctions with nonzero eigenvalues. A similar proof shows that each T_t with $t \neq 0$ is ergodic.

Suppose that $\phi \in L^2(X)$ is an eigenfunction for $\{T_t\}$ with eigenvalue $\lambda \neq 0$. This means that $\phi(T_t x) = e^{i\lambda t}\phi(x)$ for all x in X and all real t.

If A is a measurable subset of X and f is a complex measurable function defined on A, say that f equals the number c to within ϵ on A if there is a subset A' of A such that $\mu(A') > (1 - \epsilon)\mu(A)$ and $|f(x) - c| < \epsilon$ for all x in A'.

A simple approximation argument (details are in Chacon's paper [2]) shows that there is an $\eta = \eta(\phi) > 0$ such that given $\epsilon > 0$, there is a $\delta > 0$ and n_0 such that for all $n > n_0$, ϕ equals a number c, with modulus greater than η, to within $\epsilon/4$ on $[0, 2^{-n}) \times [j\delta, (j + 1)\delta)$ for some integer j between 0 and h_n/δ. It follows that ϕ equals c to within ϵ on each of the sets

$$A_k = [k2^{-n-2}, (k + 1)2^{-n-2}) \times [j\delta, (j + 1)\delta) \qquad (1 \leq k \leq 4).$$

By considering X_{n+1} and X_{n+2}, we see that

$$T_{h_n}(A_1) = A_3 , \qquad T_{h_n+d_n}(A_3) = A_2 .$$

Thus, if $\epsilon < \frac{1}{2}$,

$$|e^{i\lambda h_n}c - c| < 2\epsilon, \qquad |e^{i\lambda(h_n+d_n)}c - c| < 2\epsilon.$$

It follows that

$$|e^{i\lambda d_n} - 1| < 4\epsilon/\eta \qquad (n > n_0).$$

Since $\{d_n : n > n_0\}$ is dense in $(0, \infty)$, if $\epsilon < \eta/4$ we obtain a contradiction to $\lambda \neq 0$. Thus density of $\{d_n\}$ guarantees that $\{T_t\}$ is weakly mixing.

To show that $\{T_t\}$ is not mixing, consider the set

$$A = [0, \tfrac{1}{4}) \times [0, 1) \subset X_0 .$$

Note that A occurs in X_n as a collection of rectangles of base 2^{-n} and height 1. In passing to X_{n+1}, this pattern is replicated in both halves, and the copies are displaced by h_n. Thus

$$\mu([T_{h_n}A] \cap A) \geq \tfrac{1}{2}\mu(A).$$

Since $\mu(A) < \frac{1}{4}$, $\frac{1}{2}\mu(A) > \mu(A)^2$, so that $\{T_t\}$ is not mixing. This proof is independent of the choice of d_n.

References

1. W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Math. J. 9(1942), 25–42.
2. R. V. Chacon, *Change of velocity in flows*, J. Math. Mech. 16(1966), 417–431.
3. Nathaniel A. Friedman, *Introduction to ergodic theory*, Van Nostrand, New York, 1970.
4. Eberhard Hopf, *Complete transitivity and the ergodic principle*, Proc. Nat. Acad. Sci. 18(1932), 204–209.

Department of Mathematics, University of California, Berkeley, California 94720