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# BERNOULLICITY OF SOLENOIDAL AUTOMORPHISMS AND GLOBAL FIELDS*

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ABSTRACT

We show that ergodic automorphisms of solenoids are isomorphic to Bernoulli shifts by using the product formula for global fields.

Automorphisms of compact groups preserve Haar measure, and their ergodic properties have been intensively studied. Using Fourier approximations and Diophantine ideas, Katznelson [K] showed in 1971 that ergodic automorphisms of finite-dimensional tori are measurably isomorphic to Bernoulli shifts. This was extended first to automorphisms of infinite-dimensional tori [L1, AT], then to general compact groups, independently in [L2] and [MT]. Both proofs of the general case use Ornstein's theory of Bernoulli shifts and algebraic simplifications to reduce to the hard case, when the group is a solenoid, i.e. its dual group is a subgroup of a finite-dimensional rational vector space (for a brief account of these reductions, see [L3]). They then handle this case with different, but equally elaborate, machinery. Later Aoki [A] found a shorter proof for solenoids by extending Katznelson's arguments.

The purpose of this brief note is to observe that the arithmetical part of the proof for solenoids follows immediately from the product formula for global fields.

Let $K$ be a finite extension of $\mathbb{Q}$, and let $P$ be the set of places of $K$. The product formula states that $\prod_{v\in P} |\beta|_v = 1$ for every $\beta \neq 0$ in $K$. An excellent account of this formula is contained in Chapter 3 of [C].

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The following arithmetical result, coupled with Katznelson's Fourier approximation techniques, suffices to prove that ergodic automorphisms of solenoids are Bernoulli (see Proposition 3 of [A]).

THEOREM: *Let $\alpha$ be an algebraic number that is not a root of unity. Then there is an $n_0$ such that if $\beta \in \mathbb{Q}(\alpha)$ has the two forms*

$$(*) \qquad \beta = \sum_{j=-n^2}^{-n} c_j \alpha^j = \sum_{j=n}^{N} c_j \alpha^j,$$

*where $N \geq n \geq n_0$ and $c_j \in \mathbb{Z}$ with $|c_j| \leq |j|^{20}$, then $\beta = 0$.*

*Proof:* Let $S$ denote the set of finite places $v$ of $\mathbb{Q}(\alpha)$ for which $|\alpha|_v \neq 1$, together with all infinite places of $\mathbb{Q}(\alpha)$. If $v$ is a place of $\mathbb{Q}(\alpha)$ not in $S$, then $v$ is finite, so that $|c_j|_v \leq 1$ for all $j$ since $c_j \in \mathbb{Z}$. The ultrametric inequality applied to either sum in $(*)$ then shows that $|\beta|_v \leq 1$ for all $v \notin S$.

List the places in $S$ as $v_1, v_2, \ldots, v_q$ such that, for suitable $\theta < 1$, we have that $|\alpha|_{v_i} < \theta < 1$ for $1 \leq i \leq p$ and $|\alpha|_{v_i} \geq 1$ for $p+1 \leq i \leq q$. Note that $p \geq 1$ since $\alpha$ is not a root of unity. For $1 \leq i \leq p$, the right side of $(*)$ shows that

$$|\beta|_{v_i} \leq \sum_{j=n}^{N} j^{20} |\alpha|_{v_i}^j < C\,\theta^n,$$

where $C$ depends only on $\alpha$ and $\theta$. For $p+1 \leq i \leq q$, use of the middle term in $(*)$ shows that

$$|\beta|_{v_i} \leq \sum_{j=-n^2}^{-n} |j|^{20} \leq (n^2 - n + 1)n^{40} \leq n^{42}.$$

Recalling that $|\beta|_v \leq 1$ for all $v \notin S$, the product formula for $\mathbb{Q}(\alpha)$ shows that if $\beta \neq 0$ then

$$1 \leq \prod_{v \in S} |\beta|_v = \prod_{i=1}^{p} |\beta|_{v_i} \times \prod_{i=p+1}^{q} |\beta|_{v_i} \leq (C\,\theta^n)^p (n^{42})^q \to 0$$

as $n \to \infty$. This proves that $\beta = 0$ provided that $n$ is large enough. ∎

*Remarks:* 1. Katznelson's Diophantine result also follows directly from the product formula.

2. Let $1 < r < \theta^{-p/(q-p)}$, where $\theta, p$, and $q$ are as above. The proof shows that the Theorem remains true if the first sum in $(*)$ starts with $j = -r^n$ rather than $j = -n^2$. This exponential amount of approximate independence was shown for toral automorphisms in [L4] using other means.

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