§1. Basics

\[ R_d := \mathbb{Z}[x_1^{t_1}, \ldots, x_d^{t_d}], \quad R_2 = \mathbb{Z}[x^{t_1}, y^{t_1}] \]

\[ f(x_1, \ldots, x_d) = \sum_{n \in \mathbb{Z}_d^n} c_n x^n \in R_d \quad x^n = x_1^{n_1} \cdots x_d^{n_d} \]

\[ V_C(f) = \{ (z_1, \ldots, z_d) \in (\mathbb{C}^\times)^d : f(z_1, \ldots, z_d) = 0 \} \]

(using \( \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \) since \( f \) is Laurent)

The complex amoeba of \( f \) is

\[ A_C(f) = \{ (\log |z_1|, \ldots, \log |z_d|) : (z_1, \ldots, z_d) \in V_C(f) \} \]

\( \subset \mathbb{R}^d \)

Ex: \( d=1 \), \( f(x) = c \prod_j (x-\lambda_j) \), \( V_C(f) = \{ \lambda_j \neq 0 \} \)

\[ A_C(f) = \{ \log |\lambda_j| : \lambda_j \neq 0 \} \subset \mathbb{R} \]
Example: $d=2$, $f(x,y) = 1 + x + y$

$V_c(f) = \{(x, -1-x): x \neq 0, -1\}$

Exercise: Compute the equations of the boundary curves of $A_c(1+x+y)$

Exercise: Compute the area of $A_c(1+x+y)$

Example: $d=2$, $f(x,y) = 3 + x + y$
Ex: $f(x, y) = 5 + x + y + x^{-1} + y^{-1}$

Exercise: $0 \in A_c(f)$ Why?

Rk: If $f(x) = \sum c_n x^n$, the Newton polytope $\mathcal{N}(f)$ of $f$ is the convex hull of $\{ n \in \mathbb{Z}^d : c_n \neq 0 \}$

The tentacles of $A_c(f)$ are perpendicular to the sides of $\mathcal{N}(f)$

Rk: Can define the amoeba of an ideal $\mathfrak{a} \subset \mathbb{R}_d$ the same way:

$$A_c(\mathfrak{a}) = \{ (\log |z_1|, \ldots, \log |z_d|) : f(z_1, \ldots, z_d) = 0 \text{ all } f \in \mathfrak{a} \}$$

Then $A_c(f) = A_c(\langle f \rangle)$
§2: SOME FACTS ABOUT AMOEbas

Start with $d = 1$. When is $\mathcal{A}_d(f) = \{0\}$?

Kronecker's Thm: Suppose $f(x) \in \mathbb{Z}[x]$ is monic and all of the (complex) roots of $f$ have absolute value 1. Then all roots of $f$ are roots of unity.

Sketch of proof: Let $f(x) = \prod_{j=1}^{r} (x - \lambda_j) \in \mathbb{Z}[x]$, $|\lambda_j| = 1$ all $j$.

$$f_n(x) := \prod_{j=1}^{r} (x - \lambda_j^n) \in \mathbb{Z}[x]$$

by thm on symmetric functions. Since all $|\lambda_j| = 1$, all coefficients of $f_n$ are $\leq 2^r$ for $n = 1, 2, 3, \ldots$. So $f_n(x) = f_m(x)$ some $m \neq n$, so $\{\lambda_1^m, \ldots, \lambda_r^m\}$ is a permutation of $\{\lambda_1^n, \ldots, \lambda_r^n\}$. Conclude the $\lambda_j$'s are roots of unity.
$A_c(a) \subset \mathbb{R}^3$

$a = \langle (x^2-x-y)^2 + 1, z - 2y + x^2 - 3x \rangle$
Exercise: Make this rigorous.

2 Warning: "monic" for $f$ is crucial

Ex: $f(x) = 5x^2 - 6x + 5 \in \mathbb{Z}[x]$, roots are
\[\lambda_1 = \frac{3}{5} + \frac{4}{5}i, \quad \lambda_2 = \frac{3}{5} - \frac{4}{5}i, \quad |\lambda_1| = 1 = |\lambda_2|\]
But $\lambda_1, \lambda_2$ are not roots of unity (check this!

¿ When is $0 \in \mathcal{A}_c(f)$? This will be very significant dynamically.

Ex: $f(x, y) = 3 + x + y$ is "lopsided": one coefficient 3 is bigger than the sum of the absolute values of all others. If $0 \in \mathcal{A}_c(f)$, then $(e^{i\theta}, e^{i\phi}) \in \mathcal{V}_c(f)$, so $3 + e^{i\theta} + e^{i\phi} = 0$, but
\[3 = |3| = |1e^{i\theta} + e^{i\phi}| \leq 1 + 1 = 2 \# .

Def: $f(x) = \sum_{n} c_n x^n \in \mathbb{R}_d$ is lopsided if \[\exists n_0 \text{ so that } |c_{n_0}| > \sum_{n \neq n_0} |c_n| .\]

Rk: If $f$ is lopsided, then $0 \in \mathcal{A}_c(f)$
Rk: For a general $f$, if we can find a $g$ for which $f \cdot g$ is lopsided, then
\[
0 \notin A_c(f \cdot g) = A_c(f) \cup A_c(g)
\]
\[
\Rightarrow 0 \notin A_c(f)
\]

Thm (Kevin Purkhoo, 2006):

$0 \notin A_c(f)$ iff $\langle f \rangle$ contains a lopsided polynomial.

Rough idea when $d=1$.

\[
f(x) = \prod_{j=1}^{r} (x - \lambda_j)
\]
\[\text{no } |\lambda_j| = 1.
\]

Arrange $\lambda_j$'s as $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_q| > 1 > |\lambda_{q+1}| > \cdots > |\lambda_r|

Put $\omega = e^{2\pi i/n}$

\[
f_n(x) = \prod_{k=0}^{n-1} \prod_{j=1}^{r} (x - \omega^k \lambda_j) = \prod_{j=1}^{r} (x^n - \lambda_j^n)
\]

Then $f(x) \mid f_n(x)$ (use $k=0$ terms)

Coeff of $x^{(r-q)n}$ in $f_n(x)$ is

\[
(\lambda_1^{r-q})^n + \text{smaller stuff}
\]

and all other coeffs are also smaller.

For large enough $n$, the coeff of $x^{(r-q)n}$ will dominate.
Exercise: $d=1$, $f(x) = x^2 - x - 1$ is not lopsided, but $0 \notin \mathcal{A}_c(f) = \{ \log \left( \frac{1 + \sqrt{5}}{2} \right), \log \left( \frac{1 - \sqrt{5}}{2} \right) \}$.

Find a polynomial $g(x)$ so that $f(x)g(x)$ is lopsided.

Rk: This thm extends to ideals:

$0 \notin \mathcal{A}_c(\mathfrak{a})$ iff $\mathfrak{a}$ contains a lopsided polynomial.

§3. Nonarchimedian Amoebas

1. $1_\infty = 1_1$ is the usual absolute value on $\mathbb{Q}$.

The completion of $\mathbb{Q}$ wrt $1_\infty$ is just $\mathbb{R}$.

(and we write $\mathbb{Q}_\infty = \mathbb{R}$; $\infty$ = "infinite prime").

$p$ = prime number

Every $r \in \mathbb{Q}$ is $r = \frac{a}{b} \cdot p^m$. Define

$$v_p \left( \frac{a}{b} \cdot p^m \right) = m \quad \text{"$p$-adic valuation"}$$

$$v_p(r \cdot s) = v_p(r) + v_p(s)$$

$$v_p(r + s) \geq \min \{ v_p(r), v_p(s) \} \quad \text{(Check!)}$$

"$p$-adic absolute value on $\mathbb{Q}$" is $|r|_p = p^{-v_p(r)}$

$|r \cdot s|_p = |r|_p \cdot |s|_p$

$|r + s|_p \leq \max \{ |r|_p, |s|_p \}$ \quad \text{(ultrametric +)}
$\mathbb{Q}_p$ = completion of $\mathbb{Q}$ wrt $\| \cdot \|_p$

$r, s \in \mathbb{Q}$ are very close if $r - s$ is highly divisible by $p$

Concrete representation of $\mathbb{Q}_p$:

$\mathbb{Q}_p = \{ \sum_{n=n_0}^{\infty} c_n p^n : c_n \in \{0, 1, \ldots, p-1\}, n_0 \in \mathbb{Z} \}$

$\mathbb{Z}_p = \{ \sum_{n=0}^{\infty} c_n p^n : c_n \in \{0, 1, \ldots, p-1\} \}$

= "$p$-adic integers"

$\| \sum_{n=n_0}^{\infty} c_n p^n \|_p = p^{-n_0}$ if $c_{n_0} \neq 0$.

$\mathbb{Q}_p$ is a locally compact field

$\mathbb{Z}_p$ is a compact subring

Ostrowski's Thm: The $p$-adic absolute values $\| \cdot \|_p$ and The archimedian absolute value $\| \cdot \|_{\infty}$ are (up to a harmless power) the only absolute values on $\mathbb{Q}$. 
Rk: One consequence of ultrametric $\phi$ is:
\[ \text{if } |a|_p > |b|_p \text{ then } |a+b|_p = |a|_p \]

Hence: if $a_1 + a_2 + \ldots + a_n = 0$, then at least two $a_j$'s have the same maximal absolute value $|a_j|_p$.

**Algebraic closures**

\[ \overline{\mathbb{Q}_\infty} = \overline{\mathbb{R}} = \mathbb{C}, \quad [\overline{\mathbb{Q}_\infty} : \mathbb{Q}_\infty] = [\mathbb{C} : \mathbb{R}] = 2 \]

2 \: $\overline{\mathbb{Q}}_p$ is much more complicated: $[\overline{\mathbb{Q}}_p : \mathbb{Q}_p] = \infty$

But $1\cdot1_p$ extends to $\overline{\mathbb{Q}}_p$, and all the amoeba definitions work if we replace $\mathbb{C} = \overline{\mathbb{Q}_\infty}$ with $\overline{\mathbb{Q}}_p$.

\[ f \in \mathbb{Z}[x_1^{+1}, \ldots, x_d^{+1}] \]

\[ V_{\overline{\mathbb{Q}}_p}(f) = \left\{ (z_1, \ldots, z_d) \in (\overline{\mathbb{Q}_p})^d : f(z_1, \ldots, z_d) = 0 \right\} \]

\[ V_{\overline{\mathbb{Q}}_p}(f) = \left\{ (\log |z_1|_p, \ldots, \log |z_d|_p) : (z_1, \ldots, z_d) \in V_{\overline{\mathbb{Q}_p}}(f) \right\} \subset \mathbb{R}^d \]

$V_{\overline{\mathbb{Q}}_p}(f)$ is the "$p$-adic amoeba of $f"
Ex: \( A_\mathbb{Q}_p \)
Recall: \( 1 + x + y = 0 \Rightarrow \) at least two have same largest absolute value

\[ l_T \leq l_M \Rightarrow l_1 \leq l_M \]

Ex: \( A_\mathbb{Q}_3 \) (\( l_3 = \frac{1}{3} \))

Exercise: Compute

\[ A_\mathbb{Q}_5 \] (\( 1 + 5x + 5y + 5x^{-1} + 5y^{-1} \))

Is \( 0 \in A_\mathbb{Q}_5 \) ? [Hint: 5 is 5-adically small]
§4: Local/Global principle

The various "localizations" of \( \mathbb{Q} \) at \( p = 2, 3, 5, \ldots, \infty \) cooperate to give a global picture.

**Product Formula for \( \mathbb{Q} \):** If \( a \in \mathbb{Q}^* \) Then

\[
\prod_{p \leq \infty} |a|_p = 1 \quad [\text{Exercise: prove it!}]
\]

\[\text{Ex: } |\frac{3}{2}|_2 = 2, \quad |\frac{3}{2}|_3 = \frac{1}{3}, \quad |\frac{3}{2}|_\infty = \frac{3}{2}, \quad |\frac{3}{2}|_p = 1 \quad (p \neq 2, 3, \infty)\]

So

\[
\prod_{p \leq \infty} |\frac{3}{2}|_p = (2)(\frac{1}{3})(\frac{3}{2}) = 1 \quad \checkmark
\]

Fields for which there is a product formula are known as global fields.

The "adelic" viewpoint is to consider all local behaviour jointly together. For amoebas this means the "adelic amoeba"

\[
\mathcal{A}_\Lambda(f) := \bigcup_{p \leq \infty} \mathcal{A}_{\mathbb{Q}_p}(f)
\]
Ex: \( \sqrt{A_{A}(3+x+y)} = \)

\( \sqrt{A_{Q}(3+x+y)} \)

\( p \neq 3 \)

\( A_{Q_{3}}(3+x+y) \)

\( A_{C}(3+x+y) \)

Notice: Every ray starting at \( 0 \) hits \( A_{A}(3+x+y) \), an example of cooperative behavior.

**Thm (Einsiedler, Kapronov, L):** Let \( f \in \mathbb{Z}[x_1, \ldots, x_d] \) be non-constant. Then every ray in \( \mathbb{R}^d \) starting at \( 0 \) meets \( A_{A}(f) \).

(The first proof of this used a dynamical idea of homoclinic point!)

Recall our "counter-example" to Kronecker's Thm: \( f(x) = 5x^2 - 6x + 5 \in \mathbb{Z}[x] \) has \( A_{C}(f) = \{0\} \), but the roots \( \frac{3}{5} \pm \frac{4}{5}i \) are not roots of unity. **But**

Exercise: Show \( A_{Q_5}(f) \neq \{0\} \).
Adelic extension of Kronecker:
Let $f(x) \in \mathbb{Z}[x]$. Then

$$A_A(f) = \{0\} \text{ iff } x^n - 1 \in \langle f \rangle$$

for some $n \geq 1$.

Exercise: Prove this.

This gives a way to formulate a multi-variable version of Kronecker.

**Thm (Einsiedler, L, Ward).** Let $f(x_1, \ldots, x_d)$ be in $\mathbb{Z}[x_1^{e_1}, \ldots, x_d^{e_d}]$. Then $A_{/A}(f)$ is contained in a proper subspace of $\mathbb{R}^d$ iff there is $(n_1, \ldots, n_d) \neq 0$ such that

$$x_1^{n_1} \cdots x_d^{n_d} - 1 \in \langle f \rangle$$

This remains true if $\langle f \rangle$ is replaced by an ideal $a$ in $\mathbb{Z}[x_1^{e_1}, \ldots, x_d^{e_d}]

Problem: Is there an algorithm that decides, given a finite set $f_1, \ldots, f_k$ of generators for an ideal $a$, whether or not $a$ contains $x^n - 1$?
References for Lecture 3

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