Math 427 First Midterm Exam
November 1, 2004

Instructions: There are five problems, with the number of points for each problem indicated at the start of the problem. There is a total of 100 points.

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put the answer you want considered in the [Box] provided.
- You MUST show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.

<table>
<thead>
<tr>
<th>Problem 1</th>
<th>15 points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 2</td>
<td>20 points</td>
</tr>
<tr>
<td>Problem 3</td>
<td>30 points</td>
</tr>
<tr>
<td>Problem 4</td>
<td>20 points</td>
</tr>
<tr>
<td>Problem 5</td>
<td>15 points</td>
</tr>
<tr>
<td>Total</td>
<td>100 points</td>
</tr>
</tbody>
</table>
1. (15 points) Represent \( \frac{6}{i + \sqrt{3}} \) in polar form.

Answer:

\[
3 e^{-\frac{\pi}{6}} i
\]

\[
\frac{6}{i + \sqrt{3}} = \frac{6}{i + \sqrt{3}} \frac{\sqrt{3} - i}{\sqrt{3} - i} = \frac{6(\sqrt{3} - i)}{4} = \frac{3}{2} (\sqrt{3} - i)
\]

\[
|\sqrt{3} - i| = \sqrt{3 + 1} = 2, \text{ so } \left|\frac{3}{2} (\sqrt{3} - i)\right| = 3 = r.
\]

Argument is \(-30^\circ = -\frac{\pi}{6}\)
2. (20 points) Find all complex numbers \( z \) for which

\[ e^z (e^z - 1) = 2. \]

[Hint: There are infinitely many solutions.]

Answer:

\[ \log 2 + 2n\pi i, \quad (\pi + 2n\pi) i, \quad n = 0, \pm 1, \pm 2, \ldots \]

Let \( w = e^z \).

\[ w(w-1)=2 \Rightarrow \]

\[ w^2 - w - 2 = 0 \]

\[ \Rightarrow (w-2)(w+1)=0 \quad \text{so} \quad w=2 \text{ or } w=-1. \]

Solutions given by:

\[ e^z = 2 \text{ or } e^{x+iy} = 2, \quad \text{or} \quad e^x = 2, \quad e^{iy} = 1 \]

so all solutions here are \( x = \log 2, \ y = 2n\pi \).

\[ \log 2 + 2n\pi i \quad (n = 0, \pm 1, \ldots) \]

\[ e^z = -1, \quad \text{or} \quad e^{x+iy} = -1, \quad \text{or} \quad e^x = 1, \quad e^{iy} = -1 \]

so \( x = 0, \text{ and } y = \pi + 2n\pi, \ n = 0, \pm 1, \ldots \)

\[ \pm (\pi + 2n\pi), \ n = 0, \pm 1, \pm 2, \ldots \]
3. (30 points) Find the image in the $w$-plane of the line $\text{Re}(z) = 1$ in the $z$-plane under each of the following three mappings. For each mapping, draw a carefully labelled sketch of the image in the $w$-plane, making sure you label places where the image crosses an axis.

(a) $w = z^2$: 
\[ w = z^2 = (x + iy)^2. \]  
$\text{Re}(z) = 1$ means 
\[ w = (1 + iy)^2 = 1 + 2iy - y^2 = 1 - y^2 + 2yi. \]  
So $u = 2y$, or $\frac{\sqrt{u}}{2} = y$. Then 
\[ u = 1 - y^2 = 1 - \left(\frac{\sqrt{u}}{2}\right)^2, \] 
and the image is the curve $u = -\frac{u^2}{4} + 1$, a parabola.

(b) $w = \frac{1}{z}$: 
\[ u + iv = \frac{1}{z}, \quad z = \frac{u - iv}{u^2 + v^2}. \]  
$\text{Re}(z) = 1 \Rightarrow 1 = \frac{u}{u^2 + v^2}$, so 
\[ u^2 + v^2 = u, \] 
or 
\[ u^2 - u + v^2 = 0, \] 
\[ u^2 - u + \frac{1}{4} + v^2 = \frac{1}{4}, \] 
\[ (u - \frac{1}{2})^2 + (v - 0)^2 = \left(\frac{1}{2}\right)^2. \]  
and get the circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0).$
(c) \( w = e^z \):

\[ \text{Re}(z) = 1 \Rightarrow z = 1 + iy. \]

\[ w = u + iv = e^{1 + iy} = e \cdot e^{iy} \]

\[ |e^{iy}| = 1 \]

and this is the circle of radius \( e \) centered at the origin:

![Diagram showing a circle with radius e centered at the origin with arrows for u and e]
4. (20 points) Let \( f(z) = f(x + iy) = (x e^x \cos y - y e^x \sin y) + i(x e^x \sin y + y e^x \cos y). \) Find all points \( z = x + iy \) where \( f \) is differentiable as a complex function.

Answer: \( \text{All } z \in \mathbb{C}. \)

\( u(x,y) = x e^x \cos y - y e^x \sin y \)
\( v(x,y) = x e^x \sin y + y e^x \cos y \)

Check the CR equations: \( u_x = v_y \), \( u_y = -v_x \)

\( \frac{\partial u}{\partial y} = -x e^x \sin y - e^x \left[ y \cos y + \sin y \right] \)

\( \frac{\partial u}{\partial x} = (x e^x + e^x) \cos y - y e^x \sin y = -\frac{\partial u}{\partial y} \quad \checkmark \)

\( \frac{\partial v}{\partial x} = (x e^x + e^x) \sin y - y e^x \cos y \)

\( \frac{\partial v}{\partial y} = x e^x \cos y + e^x \left[ -y \sin y + \cos y \right] = \frac{\partial u}{\partial x} \quad \checkmark \)

All partials are continuous, CR equations are satisfied, so \( f(z) \) is differentiable at all \( z \in \mathbb{C}. \)
5. (15 points) (a) Find a complex linear function which maps the line $\text{Im}(z) = 0$ in the $z$-plane to the line $\text{Re}(w) = 1$ in the $w$-plane, and also maps the point $z = 1$ to the point $w = 1 + 3i$.

Answer:

$$f(z) = iz + 1 + 2i$$

(b) Find a second complex linear function, different from your answer to part (a), which also satisfies the mapping requirements from part (a).

Answer:

$$g(z) = 2iz + 1 + i$$

(a) $z \rightarrow iz$ rotates $90^\circ$, takes 1 to $i$. Then add $1 + 2i$, so that $1 \rightarrow 1 + 3i$. The composition is $f(z) = iz + 1 + 2i$.

(b) $z \rightarrow 2iz$ rotates $90^\circ$, takes 1 to $2i$. Then add $1 + i$, so that $1 \rightarrow 1 + 3i$. The composition is $g(z) = 2iz + 1 + i$. 