1. (25 points) Recall that a function $u(x, y)$ is called harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
(a) Using the definition, show that $u(x, y) = y^3 - 3x^2y$ is harmonic.

Answer:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

(b) Find a function $v(x, y)$ so that $u(x, y) + iv(x, y)$ is an entire analytic function.

Answer:
$$v(x, y) = -3xy^2 + x^3$$

(a) $\frac{\partial u}{\partial x} = -6xy, \quad \frac{\partial^2 u}{\partial x^2} = -6y$ \implies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0 \implies$ $u(x, y)$ is harmonic

(b) CR equations: $u_x = v_y, \quad u_y = -v_x$. So
$$u_x = v_y \implies -6xy = v_y \implies v(x, y) = \int (-6xy) \, dy + C(x) = -6x \frac{y^2}{2} + C(x) = -3xy^2 + C(x)$$
$$v_x = -u_y \implies -3y^2 + C'(x) = -3y^2 + 3x^2 \implies C'(x) = 3x^2 \implies C(x) = x^3$$

$\therefore v(x, y) = -3xy^2 + x^3$ (no need to add a constant since only one function is asked for)
2. (21 points) Find the radius of convergence for each of the following three series.

(a) \[ \sum_{n=0}^{\infty} 2^{(2^n)} z^n \] Answer: \( p = 0 \)

Let \( a_n = 2^{(2^n)} \). The ratio test gives \( p = \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} \)

Here \( \frac{a_{n+1}}{a_n} = \left( \frac{2^{(2^{n+1})}}{2^{(2^n)}} \right) = 2^{2^n} = 2 \quad \to \infty \quad \text{as} \quad n \to \infty \).

So \( \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = \infty \), and so \( p = \frac{1}{\infty} = 0 \).

(b) \[ \sum_{n=0}^{\infty} (n^2 + 17n + 2004)z^n \] Answer: \( p = 1 \)

Let \( a_n = n^2 + 17n + 2004 \). Again using the ratio test,

\[ \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = \limsup_{n \to \infty} \frac{(n+1)^2 + 17(n+1) + 2004}{n^2 + 17n + 2004} \]

\[ = \limsup_{n \to \infty} \frac{(1 + \frac{1}{n})^2 + 17\left(1 + \frac{17}{n} + \frac{2004}{n^2}\right)}{1 + \frac{17}{n} + \frac{2004}{n^2}} = 1. \quad \text{So} \quad p = \frac{1}{1} = 1 \]

(c) \[ \sum_{n=1}^{\infty} \frac{z^n}{n^n} \] Answer: \( p = \infty \)

Let \( a_n = \frac{1}{n^n} \). Using the root test,

\[ \limsup_{n \to \infty} a_n^{1/n} = \limsup_{n \to \infty} \left( \frac{1}{n} \right)^{1/n} = \limsup_{n \to \infty} \frac{1}{n} = 0, \quad \text{so} \]

\[ p = \frac{1}{\limsup_{n \to \infty} a_n^{1/n}} = \frac{1}{0} = \infty \]
3. (15 points) Find all values of \( \log(-i) \).

Answer:

\[
\log(-i) = -\frac{\pi}{2}i + 2n\pi i, \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
\log(-i) = \ln |1-i| + i \arg(-i)
\]

\[
= \ln(1) + i \left[ \arg(-i) + 2n\pi \right], \quad n = 0, \pm 1
\]

\[
= 0 + i \left[ -\frac{\pi}{2} + 2n\pi \right], \quad n = 0, \pm 1, \ldots
\]
4. (15 points) Find all values of $\pi^i$. Put your answer into rectangular form (that is, $x + iy$ form).

Answer: 

$$e^{-2n\pi e} \cos [e \cdot \ln (\pi)] + i e^{-2n\pi e} \sin [e \cdot \ln (\pi)]$$

$n = 0, \pm 1, \pm 2, \ldots$

$$\pi^i = e^{i \log (\pi)}$$

$$\log (\pi) = \ln (\pi) + 2n \pi i, \quad n = 0, \pm 1, \pm 2, \ldots$$

So $$\pi^i = e^{i \left[ \ln (\pi) + 2n \pi i \right]}$$

$$\pi^i = e^{-2n\pi e + i (e \cdot \ln (\pi))}$$

$$\pi^i = e^{-2n\pi e} \cdot e^{i [e \cdot \ln (\pi)]}$$

$$\pi^i = e^{-2n\pi e} \left\{ \cos [e \cdot \ln (\pi)] + i \sin [e \cdot \ln (\pi)] \right\}$$

$$\pi^i = e^{-2n\pi e} \cos [e \cdot \ln (\pi)] + i e^{-2n\pi e} \sin [e \cdot \ln (\pi)]$$
5. (24 points) (a) Let $C_1$ be the contour defined by the part of the parabola $y = x^2$ between the values $x = -1$ and $x = 1$, oriented with increasing values of $x$. Compute

$$\int_{C_1} zdz.$$ 

\begin{center}
\textbf{Answer:} 2i
\end{center}

\[ C_1: z(t) = t + it^2,\ -1 \leq t \leq 1, \quad \text{so} \quad z'(t) = 1 + 2it \]

\[ \int_{C_1} zdz = \int_{-1}^{1} z(t)z'(t)dt = \int_{-1}^{1} (t + it^2)(1 + 2it)dt \]

\[ = \int_{-1}^{1} [t + it^2 + 2it^2 - 2t^3]dt = 3i \left. \frac{t^3}{3} \right|_{-1}^{1} = 2i \]

(b) Let $C_2$ be the line segment between $-1 + i$ and $1 + i$, again oriented to the right. Compute

$$\int_{C_2} zdz.$$ 

[Hint: $C_1$ and $C_2$ have the same endpoints, so by Cauchy's Theorem, you should get the same answer as in part (a).]

\begin{center}
\textbf{Answer:} 2i
\end{center}

\[ C_2: z(t) = t + i,\ -1 \leq t \leq 1. \quad z'(t) = 1 \]

\[ \int_{C_2} zdz = \int_{-1}^{1} (t+i)(1)dt = i \left. \right|_{-1}^{1} = 2i \]