OPEN QUESTIONS REGARDING THE POSITIVE MASS THEOREM AND THE INTRINSIC FLAT DISTANCE

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In 1979 Schoen and Yau proved the Positive Mass Theorem: *If M^3 is a complete asymptotically flat Riemannian manifold with nonnegative scalar curvature, then $m_{ADM}(M^3) \geq 0$. Furthermore, if $m_{ADM}(M^3) = 0$ then M^3 is isometric to Euclidean space.*

Their theorem leads naturally to the following open question:

If M^3 only has small ADM mass,
in what sense is it close to being Euclidean space?

Background for this question may be found in a joint paper with Dan Lee available on the ArXiV. In that paper we provide examples demonstrating that, even in the rotationally symmetric sense, M^3 is not close to Euclidean space in the smooth or Gromov-Hausdorff sense. We conjecture:

Small Positive Mass Conjecture: *Suppose M_j^3 is an asymptotically flat Riemannian manifold with no interior minimal surfaces, nonnegative scalar curvature and either $\partial M = \emptyset$ or ∂M is a minimal surface. Fixing an $\alpha_0 > 0$ and $D > 0$, if $m_{ADM}(M^3) \to 0$ then*

$$d_{\mathcal{F}}(T_D(\Sigma_{\alpha_0}) \subset M_j^3, T_D(\Sigma_{\alpha_0}) \subset \mathbb{E}^3) \to 0.$$

Recall that the intrinsic flat distance, $d_{\mathcal{F}}$, estimates the distances between Riemannian manifolds by filling in the space "between" them with a filling manifold and measuring the volume of the filling manifold and the excess boundary. This distance was introduced in [S—Wenger, JDG 2011] applying work of Ambrosio-Kirchheim [Acta-2000].

Dan Lee and I prove the Small Positive Mass Conjecture in the case where M_j^3 are rotationally symmetric manifolds. Without rotational symmetry, one may have many more wells of arbitrary depth becoming arbitrarily dense [Lee-S—, ArXiV]. Such a sequence has no Gromov-Hausdorff converging subsequence. However we believe that the total volumes of the wells will remain bounded and the Small Positive Mass Conjecture holds.

Below we provide two methods towards proving the Small Positive Mass Conjecture along with special cases and open questions which would be useful towards a proof of this conjecture and elsewhere. We also state more general conjectures regarding sequences of manifolds with positive mass and their intrinsic flat limits.

0.1. **Method I: Explicit Fillings.** One may try to prove the Small Positive Mass Conjecture with explicit fillings. The paper on the arxiv with Dan

Lee has theorems regarding the construction of filling manifolds which may prove useful in this direction.

One may wish to investigate one of the following two special cases:

1) M_j have smooth CMC foliations
   and Σ_{α_0} is a leaf of area α_0
2) M_j have smooth inverse mean curvature flows
   and Σ_{α_0} is a level of the flow of area α_0.

In general, one might try applying Huisken-Ilmanen's Inverse Mean Curvature Flow and bound the volumes of the skipped regions if possible.

0.2. **Method II: Prove the Positive Mass Theorem on Limit Spaces.** One may try to prove the Small Positive Mass Theorem by applying Wenger's Compactness Theorem to obtain limit, $M_j \xrightarrow{\mathcal{F}} M_\infty$, and then prove the Positive Mass Theorem holds on the limit space.

In joint work with Wenger [JDG-2011], we show the limit spaces achieved under intrinsic flat convegence are "integral current spaces". These spaces are also 3 dimensional and they have disjoint biLipschitz charts and are oriented with boundary. I am currently working to define nonnegative scalar curvature and positive mass on integral current spaces.

One may work to define ingredients needed to imitate one of the proofs of the Positive Mass Theorem on integral current spaces:

1) Schoen-Yau's proof needs minimal surfaces.
2) Witten's proof needs spinors.
3) Huisken-Ilmanen's proof needs inverse mean curvature flow.
4) Huisken has recent ideas involving isoperimetric domains.

These ingredients could also be useful in other applications of intrinsic flat convergence to general relativity.

0.3. **Limit Spaces need not be Connected.** There are examples constructed in 3 dimensions with positive scalar curvature that are disconnected in the limit. They may even have countably many components. [S—Wenger JDG 2011 Appendix]. These examples have interior minimal surfaces.

Conjecture: *Intrinsic flat limits of manifolds with nonnegative scalar curvature and no interior minimal surfaces are connected.*

Note that intrinsic flat limits of manifolds with nonnegative Ricci curvature are connected [S—Wenger, CalcVarPDE].

0.4. **Cancellation under Intrinsic Flat Convergence.** A sequence of manifolds M_j may converge to the 0 space in the intrinsic flat sense if one of the following occurs:

(1) $vol(M_j) \to 0$.
(2) $dim_H(Y) = 0$ where $d_{GH}(M_j, Y) \to 0$.
(3) there is cancellation as two sheets
    come together with opposite orientations:

Cancellation may occur for M_j^3 with positive scalar curvature and interior minimal surfaces [S—Wenger CalcVarPDE2010]. There we prove that cancellation does not occur in situations where filling volumes of spheres are well controlled. In particular there is no cancellation when the sequence has a uniform lower bound on Ricci curvature and volume.

Conjecture: *If M_j^3 have nonnegative scalar curvature and no interior minimal surfaces and a uniform lower bound on volume, then there is no cancellation in the limit.*

This conjecture is stated in more detail in the joint papers with Stefan Wenger.