Open Problems
Three Dimensional Manifolds All of Whose Geodesics Are Closed

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We present some open problems related to the talk given at the PNGS meeting November 7-8th at Portland State University in Portland, OR.

1 Infinitely many Closed Geodesics

The first problem is concerned with the existence of closed geodesics on a manifold (M, g) for an arbitrary metric g.

The theorem of Gromoll and Meyer states that a simply connected compact manifold M has infinitely many closed geodesics if the Betti numbers, $b_i(\Lambda M; \mathbb{F})$, of the free loop space ΛM form an unbounded sequence. If the field is the rational numbers this is by Sullivan and Vigue equivalent to the $H^*(M; \mathbb{Q})$ not being generated by one element. In particular, the theorem does not include the manifolds S^n, $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$. Hingston and Bangert showed that any metric on S^2 has infinitely many closed geodesics. In fact, for the spaces not covered by the theorem of Gromoll and Meyer and S^2, the only thing known in general is the existence of one closed geodesic.

Problem 1. *Does there exist infinitely many prime closed geodesics for any metric on S^n, $n \geq 3$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$? One could more generally ask the same questions for any compact, simply connected manifold whose Betti numbers of the free loop space form a bounded sequence.*

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2 The Berger Conjecture

The Berger states that for a simply connected manifold all of whose geodesics are closed, the geodesics have the same least period. The theorem of Bott and Samelson states that a simply connected manifold M^n all of whose geodesics are closed has the integral cohomology ring $H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/x^{n+1}$. In 1982 Grove and Gromoll proved the conjecture for metrics on S^2 and in 2009 Wilking proved it for metrics on S^n, $n > 3$. The conjecture is still open for metrics on S^3 and for metrics on manifolds with the integral cohomology ring of $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$.

Problem 2. *Settle the conjecture in the remaining cases.*

A natural approach is to assume the existence of geodesics shorter than the common period and then try to derive a contradiction to the known (equivariant) cohomology of ΛM using Morse theory on the free loop space of the manifold. For metrics on S^3, the author was able to derive some conclusions about the Morse theory of the energy function. So far nothing is known about the Morse theory of the energy function on projective spaces for general metrics all of whose geodesics are closed, and a very useful first step would be to conclude something about that.

Problem 3. *Investigate the Morse theory of the energy function on the free loop space of M for manifolds all of whose geodesics are closed.*

If one wants to approach the conjecture using Morse theory, the more detailed knowledge one has about the (equivariant) topology of the free loop space, the better. There are strong splitting results (Carlson-Cohen) for the S^1-Borel construction of ΛS^n, and it would be very interesting to have similar results for the free loop space of the projective spaces and for other groups $G \subseteq O(2)$. Many aspects of the topology of free loop spaces of spheres and projective spaces are unknown, so let us state the problem as follows.

Problem 4. *Obtain a better understanding of the (equivariant) topology of the free loop space of spheres and projective spaces.*