Isoperimetric type problems in Riemannian manifolds

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We collect a few open questions in the study of isoperimetric type problems in Riemannian manifolds.

The isoperimetric problem : In the classical isoperimetric problem, one is looking for a minimizer of the area functional subject to some volume constraint. This provides a way to construct constant mean curvature hypersurfaces since, when solutions exist and are regular enough, they are characterized by the fact that they are constant mean curvature hypersurfaces. Few information on the solutions of the isoperimetric problem are available even though some progress has been done when the ambient manifold has some special structure.

The construction of constant mean curvature hypersurfaces in Riemannian manifolds remains a hard problem.

It is very likely that hypersurfaces with large constant mean curvature have some nice structure when the ambient manifold is generic (for example when there are no Killing vector field) [5]. Examples which are available so far show that critical points of the scalar curvature and minimal submanifolds play an important role in the compactification of the space of all constant mean curvature hypersurfaces (with mean curvature constant but not fixed) [7], [4], [2], [1].

For example, if p is fixed such that there exists a sequence of constant mean curvature hypersurfaces, with mean curvature tending to infinity, which converges (for the Hausdorff distance) to the point p. Is it true that p is a critical point of the scalar curvature function ? All known examples give some credit to the fact that the answer to this question should be positive [7], [4].

Sequence of hypersurfaces with large constant mean curvature condensing on a minimal submanifold have been constructed by Mahmoudi, Mazzeo and Pacard [2], [1]. It is still an open question to construct sequence of constant mean curvature hypersurfaces which condensate on a minimal network.

Extremal domains for the first eigenvalue of the Laplacian : A related problem is the existence of extremal domains for the first eigenvalue of the Laplacian. This time, instead of minimizing the area, one minimizes the first eigenvalue of the Laplacian, with 0 Dirichlet boundary condition, among domains whose volume is prescribed. This problem is related to the existence of *extremal domains*, namely domains for which there exists a positive solution of $\Delta_g u + \lambda\, u = 0$ which vanish on the boundary and have constant Neumann data.

An existence result for extremal domains had recently been obtained by Pacard and Sicbaldi [3]. In the case where the ambient manifold is the Euclidean space, the existence of noncompact extremal domains is an interesting problem which shares the features of the study of noncompact constant mean curvature surfaces. In particular, it is known that there exists (non compact) extremal domains in Euclidean space whose boundary looks like Delaunay surfaces.

Domains for which there exists a positive harmonic function with 0 Dirichlet boundary data and constant Neumann data play, for extremal domains, the role played by minimal hypersurfaces in the theory of constant mean curvature hypersurfaces. In particular, the former arise in the blow up analysis of extremal domains. It would be interesting to classify all such domains in Euclidean space.

Références

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