Open Problems for PNGS

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1. As we will see, most separatrices of the Ricci flow on 3D unimodular Lie groups arise from extra symmetries, as with the Berger spheres. However, there is one orbit through $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ whose forward limit is the Euclidean metric on $E(2)$ and whose backward limit is the soliton metric on Sol (under a certain rescaling). What is the geometric significance of this orbit? It is interesting to note that there is an orbit of cross curvature flow with the same endpoints which is NOT the same orbit (see [CSC-08]), which indicates that it probably does not arise out of an extra symmetry of the metrics.

2. It is a long standing problem to analyze the stability of fixed points of the Ricci flow. There has been some recent work on stability of metrics on bundles by D. Knopf [K-07], which was used by Lott [L-07b] to classify the behavior of Type-III solutions of Ricci flow in three dimensions. It is interesting to compare the results of our analysis with that of Guenther-Isenberg-Knopf [GIK-06]. They found that, under compactly supported variations, Nil and Sol solitons are linearly stable. The pictures of Nil and Sol in the space of 3D unimodular Lie groups (or a linear stability analysis of the dynamical system) shows Nil to be an unstable fixed point and Sol to be a saddle point in this space. The result in [GIK-06] only claims linear stability and not dynamic stability, but the strengthening of their result to dynamic stability is not ruled out by our results because variations in the space of 3D unimodular Lie groups are not compactly supported. One can add to the difficulty how to encapsulate variations which result in compact homogeneous spaces, potentially from the Riemannian groupoid perspective (See [L-07a], [G-08]). Is it possible to get a complete understanding of the stability of Nil, the Sol soliton, and other soliton metrics in the space of Riemannian metrics modulo rescaling and diffeomorphisms?

3. It is still not entirely clear the best way to describe the space of Riemannian metrics modulo rescaling and diffeomorphisms in such a way that there is a reasonable closure of Ricci flows. The Riemannian groupoid formalism allows a closure of Type-III solutions, but it requires choosing a particular scaling. It would be nice to have a space which is independent of this choice. The picture is quite complicated, however. For instance, if

we try to consider a closure with respect to Gromov-Hausdorff distance, we may find elements in the closure which are no longer Riemannian, but sub-Riemannian as described in [CSC-08]. Different choices of rescaling certainly result in different limits (e.g., we may always choose a rescaling which limits to Euclidean space, and the limits in [IJ-92] are different than the ones in [L-07a] and [G-08]), but it may be possible to look for a framework in which all of these limits are in the same equivalence class, or in which we can argue that one choice of rescaling is certainly better than every other.

4. It was conjectured by R. Hamilton that the Ricci flow takes twisted bundles and "straightens them" and that it takes non-homogeneous manifolds and makes them more homogeneous. We can see that, for instance, the special metrics on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ which are Riemannian submersions over $\mathbb{H}^2$ do, in fact, turn a twisted bundle into a product bundle. Furthermore, Lott's work [L-07a] shows that three dimensional Type-III solutions do become more homogeneous. To what extent can we prove or disprove these assertions? Is it possible to find a quantity which measures the twistedness of a bundle (like one of its second fundamental form tensors) or homogeneity of a space and show that Ricci flow "improves" this quantity?

References

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