MANIFOLDS WITH POINTWISE 1/4-PINCHED CURVATURE

RICHARD SCHOEN

Related Problems

The most obvious remaining problem after [1] and [2] is the classification of PIC manifolds. A conjecture about PIC manifolds for which there is substantial evidence is the following.

Conjecture 1. *Let M be a compact Riemannian manifold which is PIC. There exists a finite covering $\hat{M}$ of M which is diffeomorphic to a connected sum of a finite number of copies of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.*

It is known [7] that the PIC condition is closed under the connected sum operation, and that the product on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ is PIC. The conjecture is proven in [5] for $n = 4$ (using Ricci flow with surgeries) under the additional assumption that the manifold contains no embedded incompressible three dimensional spherical space forms. Recall that it is shown in [6] that a simply connected PIC manifold is homeomorphic to $\mathbb{S}^n$. A corollary of the conjecture above is that the fundamental group of a PIC manifold should have a finite index subgroup which is a free group. Strong restrictions on the fundamental group which support this conclusion were given in [3] and [4]. The results aside from those of [5] use the variational theory for minimal surfaces. A very interesting feature of the PIC condition is that there are two parallel methods of attack which are not clearly related in any technical way. This is reminiscent to the study of positive scalar curvature which can be attacked by either the Dirac operator or minimal hypersurface theory. While there is substantial overlap in the results that have been derived from the two approaches, there is not clear map between them. It would be useful to understand the relationship between the minimal surface and Ricci flow methods especially since a combination of them could lead to stronger results.

There are many open problems for manifolds of positive curvature. The most important seems to be the problem of distinguishing manifolds of nonnegative curvature from those of positive curvature. For example, the following version of a conjecture of H. Hopf seems to be a central question.

Conjecture 2. *A compact symmetric space of rank greater than* 1 *cannot carry a Riemannian metric of positive curvature.*

Recall that any compact symmetric space has nonnegative curvature, but it is unknown whether there are any symmetric spaces of rank greater than 1 which also have Riemannian metrics of positive curvature. The simplest special case of this conjecture would be to show that $\mathbb{S}^2 \times \mathbb{S}^2$ cannot carry a metric of positive curvature. There seems to be a lack of available technique for approaching this question. The Ricci flow approach which works for 1/4-pinched manifolds does not appear to be helpful for this question.

A second famous Hopf conjecture is the following.

Conjecture 3. *An compact manifold of nonnegative curvature has nonnegative Euler characteristic.*

This is a beautiful question which has a positive answer for the compact symmetric spaces, but the lack of examples at present makes the evidence for it rather weak.

References

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Department of Mathematics, Stanford University, Stanford, CA 94305