OPEN PROBLEMS

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1. The volume of the moduli space of quintics. The volume of moduli space contains geometric and physical information of the original polarized manifold. In the following, we propose to compute the volume of quintic moduli.

Let the quintic hypersurface in CP^4 be

$$X = \{Z \mid Z_0^5 + \cdots + Z_4^5 + 5\lambda Z_0 \cdots Z_4 = 0\} \subset CP^4.$$

It is a smooth hypersurface if λ is not any of the fifth unit roots. To construct the moduli space, we define

$$V = \{f \mid f \text{ is a homogeneous quintic polynomial of } Z_0, \cdots, Z_4\}.$$

We can verify that $\dim V = 126$. Thus for any $t \in P(V) = CP^{125}$, t is represented by a hypersurface. However, if two hypersurfaces differ by an element in $Aut(CP^4)$, then they are considered to be the same. Let D be the divisor in CP^{125} characterizing the singular hypersurfaces in CP^4. Then the moduli space of X is

$$\mathcal{M} = CP^{125} \backslash D / Aut(CP^4).$$

The dimension of the moduli space is 101.

The physics background of the question is explained in the paper [3] and the references in that paper.

The observation is that the volume can be computed via the computation of the volume in the Hilbert scheme of $\mathcal{M}$, which in this case is the CP^{125}. However, the difficulty we have to overcome is the control of the WP metric near the discriminant divisor that characterizing the singular quintic hypersurfaces. Partial result in resolving growth near the discriminant divisor was obtained (by Douglas and Lu).

2. Rigidity and ergodicity of CY moduli. Motivated by the superrigidity of Margulis, it is natural to conjecture that in some sense, the monodromy group (or the fundamental group) determines the CY moduli. One of the evidence is the following example:

The moduli space of an algebraic $K3$ surface is [5, 1, 2]

$$SO(2, 19, \mathbb{Z}) \backslash SO(2, 19)/S(O(2) \times O(19)),$$

which is a Hermitian symmetric space of rank 2. For such a space, we do have the (Margulis) supperrigidity. What is the rigidity theorem in the case of CY moduli?

To be more precise, we make the following two conjectures:

Conjecture 1. Let (X_1, L_1) and (X_2, L_2) be two polarized CY manifolds and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the corresponding moduli spaces. Suppose Γ_1 and Γ_2 are the monodrmy groups

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of the two moduli spaces. Suppose that there is an isomorphism

$$\iota : \Gamma_1 \to \Gamma_2$$

in the sense of abstract groups. Then there is a holomorphic isometry (with respect to the WP metric or the Hodge metric)

$$f : \tilde{M}_1 \to \tilde{M}_2,$$

which is Γ_1-equivariant, where $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$ are the universal covering spaces of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively.

Conjecture 2. Given a fixed CY moduli space $\mathcal{M}$. We emphasis that $\mathcal{M}$ is connected. If (X_1, L_1) and (X_1, L_2) are two CY manifolds such that $\mathcal{M}$ is the moduli space of both of them. Then there is a path σ_t connecting X_1 and X_2 in $\mathcal{M}$.

Both conjectures are difficult, because the curvature properties of the WP and the Hodge metrics are not enough for the existence of harmonic maps. On the other hand, CY moduli is always quasi-projective [6] and the volume is always finite [4]. Thus it seems that the first step toward proving the conjecture is the following:

Conjecture 3. CY moduli is always a concave manifold.

By concavity, we mean that there exists an exhaustion function on the manifold such that at each point the (complex) Hessian form has at least two negative eigenvalues.

3. Incompleteness of the Weil-Petersson metric. For all known examples, the CY moduli are WP incomplete. However, mathematically proving the incompleteness of the metric is very important because it is related to the following conjecture in algebraic geometry

Conjecture 4. Let X be a non-rigid primitive [1] Calabi-Yau threefold. Then it can be degenerated to a singular Calabi-Yau variety with one ODP.

To see the relations between the above conjecture and the incompleteness of the WP metric, we recall the following fact [7]

Theorem 1 (Wang)**.** *Let $\mathcal{X} \to \Delta^*$ be a one-parameter family of smooth Calabi-Yau manifold. Suppose that the center fiber is a variety with isolated terminal singularities. Then the WP metric is incomplete.*

Thus proving the incompleteness of CY moduli will give a convincing evidence of Conjecture 4.

References

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