Open Problems

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In general one may ask how much of Theorem 1.3 in [H99] can be generalized to dimension bigger than three, in particular dimension 4. A couple of more specific questions are given below.

Let M be a closed manifold. Suppose $(M^n, g(t)), t \in [0, \infty)$ is a solution of Ricci flow and $\hat{g}(\hat{t})$ is the corresponding volume-normalized solution. Define

$$\hat{\imath}_M(\hat{t}) = \max_{x \in M} injrad_{\hat{g}(\hat{t})}(x)$$

$$\tilde{V}_\infty = \lim_{t \to \infty} \frac{\int_M dg(t)}{t^{n/2}},$$

where $injrad_{\hat{g}(\hat{t})}(x)$ is the injectivity radius of $\hat{g}(\hat{t})$ at x.

(1) Suppose $\lim_{\hat{t} \to \infty} \hat{\imath}_M(\hat{t}) = 0$, show that there is a constant $C > 0$ such that $\left|Rm_{g(t)}(x)\right| \le \frac{C}{t}$ for all x, t and that $(M, \hat{g}(\hat{t}))$ collapse to a lower dimensional Euclidean space. In particular this should be true when $\hat{g}(\hat{t})$ is a homogeneous metric and M is a Lie group. [IJL] shows that the last statement is true in dimension 4 and $\hat{g}(\hat{t})$ is of certain diagonal form.

(2) Assume $\tilde{V}_\infty > 0$, show that there is an open set $U \subset M$ and $t_j \to \infty$ such that

(2a) $g(t_j)/t_j \to \tilde{g}_\infty$ on U;

(2b) $\tilde{g}_\infty$ is an Einstein metric with negative scalar curvature;

(2c) $Vol(U, \tilde{g}_\infty) = \tilde{V}_\infty$ (see [FIN] for more details).

References

[H99] R. Hamilton, *Non-singular solutions of the Ricci flow on three-manifolds,* Comm. Anal. Geom. **7** (1999) 695-729.

[FIN] M. Feldman, T. Ilmanen and L. Ni, *Entropy and reduced distance for Ricci expanders*, math.DG/0405036.

[IJL] J. Isenberg, M. Jackson and P. Lu, *Ricci flow on locally homogeneous closed 4-manifolds,* math.DG/0502170.