PACIFIC NORTHWEST GEOMETRY SEMINAR

OPEN PROBLEMS, WINTER 2004

Jim Bryan: Computing the closed topological vertex via the geometry of the Cremona transform

Problem 1: Develop a mathematical theory of the "open vertex" technology from physics that computes the Gromov-Witten invariants of any toric Calabi-Yau threefold.

Problem 2: Find the mathematics behind the Chern-Simons/string theory duality. In particular, find a direct connection between quantum invariants of knots and Gromov-Witten theory of Calabi-Yau threefolds.

Problem 3: Prove the Donaldson-Thomas/Gromov-Witten correspondence that has been conjectured by Maulik-Nekrasov-Okounkov-Pandharipande.

Problem 4: "Symplecticize" Donaldson-Thomas theory. i.e. find a gauge theoretic construction of the Donaldson-Thomas invariants that applies in the almost Kahler category.

Problem 5: Develop a gluing theory for Donaldson-Thomas invariants. That is, find the analogue in Donaldson-Thomas theory (expected to exist from the GW/DT correspondence) of Symplectic Field Theory and/or relative stable maps in Gromov-Witten theory.

Jon Wolfson: Homology, Holomorphicity and Area Minimizers in K3 surfaces

Problem 1

This is a technical question.

Let X be a K3 surface. Suppose that g is a Calabi-Yau metric and Σ_1 and Σ_2 are -2-curves (imbedded rational curves) that intersect orthogonally at a point p. Is there a deformation of g, through Calabi-Yau metrics fixing the complex structure, so that for the perturbed metrics that intersection is non-orthogonal? The deformation space of g, fixing the complex structure, can be identified with the g-harmonic $(1,1)$-forms on X. Therefore the question can be reintrepreted as a question about the values of these forms at p.

Problem 2

Let X be a K3 surface and g a Calabi-Yau metric. Let $\alpha \in H_2(X, \mathbf{Z})$ and denote the self intersection of α by $\alpha \cdot \alpha$. In joint work with M. Micallef [M-W] (described in the talk) we consider the following problem: Given a Calabi-Yau metric g and an integral homology class $\alpha \in H_2(X, \mathbf{Z})$ is an area minimizer of α given by a sum of surfaces each calibrated by some Kähler form compatible with g? We note that if $\alpha \cdot \alpha \geq -2$ then every area minimizer is calibrated by a Kähler form compatible with g. Also every class $\alpha \in H_2(X, \mathbf{Z})$ can be represented by a sum of surfaces each calibrated by some Kähler form compatible with g. However, we show that there is a class $\alpha \in H_2(X, \mathbf{Z})$ and a Calabi-Yau metric g so that no area minimizer decomposes into a sum of surfaces each calibrated by some Kähler form compatible with g. In our construction the metric g is near the boundary of the space of Calabi-Yau metrics. That is, it is close to an orbifold metric.

Question: Given $\alpha \in H_2(X, \mathbf{Z})$ with $\alpha \cdot \alpha < -2$ describe the Calabi-Yau metrics such that an area minimizer of α is given by a sum of surfaces each calibrated by some Kähler form compatible with g. In particular is this set non-empty?

References

[M-W] Micallef, M. and Wolfson, J., Area minimizers in a K3 surface and holomorphicity, preprint.

Charles Boyer: Einstein Metrics on Exotic Spheres

• **Conjecture:** All homotopy spheres which bound parallelizable manifolds admit Sasakian-Einstein metrics.

• Let $K_{\min}$ denote the minimal value of the sectional curvature. Can one obtain an estimate for $K_{\min}$? or better a formula in terms of the weights and degree of the BP polynomial?

• Is there a bound on the dimension of the moduli space of Sasakian-Einstein metrics on a given manifold, (or more generally for appropriately normalized Einstein metrics) on a given manifold? Find formulae that depend only on dimension. Recall that S^{13} has a moduli space of Sasakian-Einstein metrics of dimension greater than 2.1×10^{13}.

• How is the moduli space of Sasakian-Einstein metrics related the full moduli space of Einstein metrics?

• Does the moduli space of Sasakian-Einstein metrics have an infinite number of components? This is true for the moduli space of deformation classes of positive Sasakian structures on spheres [B,Galicki,Nakamaye] (Topology 42 (2003) 981-1002), using work of Morita and Ustilovsky on distinct contact structures. However, our proofs of Sasakian-Einstein metrics on spheres only yield a finite number of deformation classes.

• Does the dimension of a component depend on $K_{\min}$?