PACIFIC NORTHWEST GEOMETRY SEMINAR

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Michael Anderson: Surgery construction of Einstein metrics

1. The PGNS talk described how Dehn filling or surgery on the cusps of hyperbolic manifolds gives Einstein metrics, whenever the filling is sufficiently large. Are there other examples of surgery that preserve the Einstein condition?

2. Can one develop an existence theory for the solution to the Dirichlet problem for Einstein metrics on smooth compact manifolds with boundary? Progress has been made in this situation in the complete case, when the boundary is at infinity, but the situation of finite boundary should be easier, and of interest.

3. There are curves of Einstein metrics on certain closed 4-manifolds which limit not on an Einstein metric on the original 4-manifold, but on "cusp-type" Einstein metrics. The simplest example is to take the manifold as a product of surfaces of higher genus, and a curve of product hyperbolic metrics, where either or both factors tend to the boundary of hyperbolic space. It would be very interesting to know if such curves which limit on cusp-type metrics exist, or do not exist, in the context of complete non-compact Einstein metrics which are conformally compact, or asymptotically hyperbolic. This would have important consequences regarding the Dirichlet problem, mentioned in Problem 2.

Christine M. Escher: The topology of positively curved manifolds

Eschenburg spaces. Complete the classification of Eschenburg spaces, i.e. compute all topological invariants for those spaces that do not satisfy condition (D) of Kruggel [Kr2] .

Einstein metrics. Determine exactly which of the Eschenburg spaces admit Einstein metrics. The same question can also be asked for the generalized Witten manifolds $M_{k,l}$ which are defined as orbit spaces of the following S^1-action on $S^5 \times S^3$:

$$\begin{aligned} S^1 \times S^5 \times S^3 &\longrightarrow S^5 \times S^3 \\ (z, (u_1, u_2, u_3), (v_1, v_2)) &\longmapsto (z^k u_1, z^k u_2, z^k u_3), (z^{l_1} v_1, z^{l_2} v_2)) \,, \end{aligned}$$

where $S^5 \subset \mathbb{C}^3$, $S^3 \subset \mathbb{C}^2$ and k and l_j are non-zero integers for $j \in \{1, 2\}$. This action is free if and only if $\gcd(k\,,\, l_j) = 1$ for $j \in \{1, 2\}$. Hence one obtains the *generalized Witten manifolds* $N_{k,l} := (S^5 \times S^3)/S^1$ where $l = (l_1\,, l_2)$ and $\gcd(k\,,\, l_j) = 1$ for $j \in \{1, 2\}$. The topological classification of these spaces is given in [E].

Other Families with $K \geq 0$. Give a topological classification of some of the new examples of nonnegative sectional curvature as given in [GZ]). In particular, classify the principal $S^3 \times S^3$- bundles over S^4.

Generalizations to dimensions 9 and 11. Various of these families of manifolds of nonnegative curvature have generalizations to dimensions bigger than 7. For example, the family $P_{k,l}$ of circle bundles over $\mathbb{C}^n \times \mathbb{C}^m$, which can also be described as orbit spaces of S^1-actions on $(S^{2n+1} \times S^{2m+1}$ by

$$\begin{aligned} S^1 \times S^{2n+1} \times S^{2m+1} &\longrightarrow S^{2n+1} \times S^{2m+1} \\ (z, (X, Y)) &\longmapsto (z^k X, z^l Y)\,, \end{aligned} \tag{1}$$

using the standard S^1-action on S^{2n+1} and S^{2m+1} .

References

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Victor Guillemin: Signature quantization

The main unsolved problem in this subject is to show that signature quantization is completely characterized by additivity with respect to the basic cutting and gluing operations in symplectic geometry. This problem, however, is a bit ill-defined since it isn't one hundred percent clear what one means by "quantization". I proposed at the end of my talk one possible definition: Take a "Dirac-type" operator, D, and twist it by the prequantum line bundle. (We're assuming that the symplectic manifold in question is pre-quantizable, so it has a line bundle with connection whose curvature form is the symplectic form. This is the pre-quantum line bundle.) Now take the quantization of M to be the virtual vector space consisting of the kernel minus the cokernel of D^L. This gives, by the way a lot of "quantizations" since there are a lot of candidates for D.

Rafe Mazzeo: Positive Scalar Curvature and Poincaré-Einstein Fillings

(1) Does there exist a sequence of Poincare-Einstein (conformally compact Einstein) metrics on the ball B^{n+1} which develop cusp singularities?

(2) The theme of my talk was whether it is possible to find Poincare-Einstein metrics which bound an arbitrary scalar positive conformal class $[h]$ on a 3-manifold Y. I presented some substantial progress towards this, joint work with Michael Singer, but the general case remains open. Specifically, if Γ is an arbitrary finite subgroup of $U(2)$ which acts freely on the sphere S^3, then does the space-form S^3/Γ bound such a (smooth) metric? At present we can only answer this in some cases.

(3) The degree theory for the Poincare-Einstein filling problem in four dimensions developed by Anderson needs more careful development; also, describe the behaviour of this Anderson degree under the boundary connect sum construction of Mazzeo-Pacard. We conjecture that the degree is multiplicative in this construction.

Matthew Gursky:
A notion of maximal volume in conformal geometry and some applications

1. Despite the obvious relation between the so-called σ_k-Yamabe problem–that is, conformally deforming a metric so that the $k-$th elementary symmetric polynomial of the Schouten tensor is constant–and the classical Yamabe problem, there are some important technical and geometric differences. For example, when the Schouten tensor of the background metric is in the "negative cone" (for example, when it is negative definite), there is no existence theory. While the C^0- and C^1-estimates were established by Viaclovsky, his arguments (which involve the maximum principle) break down at the C^2-level. This raises several questions:

(a) Can one construct (even a local) counter-example to the C^2-estimate? That is, can one construct a sequence of conformal metrics $g_k = e^{2u}g$ for which $\sigma_k(A)$ is converging to a constant, but $\|u\|_{C^2} \to \infty$? (Of course, we are assuming here that $k \leq 2$).

(b) Assuming the C^2-estimate fails in general, is there a geometric obstruction? That is, does it fail because there is an obstruction to admitting a metric with constant $\sigma_k(A)$? Since *every* compact manifold admits a metric of constant scalar curvature, this would be surprising.

2. Turning to more global questions, the work of Guan-Viaclovsky-Wang showed that a metric in the positive $k-$cone with $k > n/2$ has positive Ricci curvature. So under what conditions can one conformally deform a metric so that it ends up in such a cone? The work of Guan-Lin gave certain (conformally invariant) conditions which are sufficient for the existence of such a deformation; but their work was in a rather special setting (locally conformally flat manifolds). In this case, the manifold is conformally covered by the round sphere; thus it would be interesting to find a more general condition.

Think of this question as asking: "When is a metric conformal to one of positive Ricci curvature?"

3. The existence work of Gursky-Viaclovsky introduced the notion of "maximal volume" as a variational formulation of studying the σ_k-- Yamabe problem. But so far, this quantity is only understood in low dimensions. Any sharp estimate in higher dimensions would be quite interesting, as the arguments in three and four dimensions are completely unrelated.