Math 549 **Complex Manifolds** **Spring 2019**

ASSIGNMENT #2
DUE 5/10/19

1. Let (M, J) be an almost complex manifold, and define $\Lambda^{p,q}M$ just as we did for complex manifolds. Show that the following are equivalent:
   (a) J is integrable.
   (b) For each pair of nonnegative integers p, q, the exterior derivative d maps smooth sections of $\Lambda^{p,q}M$ to sections of $\Lambda^{p+1,q}M \oplus \Lambda^{p,q+1}M$.
   (c) d maps sections of $\Lambda^{0,1}M$ to sections of $\Lambda^{1,1}M \oplus \Lambda^{0,2}M$.
2. THE LOCAL $\partial\overline{\partial}$-LEMMA: Suppose ω is a smooth, real, closed $(1,1)$-form on a complex manifold M. (To say that ω is ***real*** just means that $\overline{\omega} = \omega$.) Prove that in a neighborhood of each point of M, there exists a real-valued smooth function u such that $\omega = i\partial\overline{\partial}u$.
3. Let $H \to \mathbb{CP}^n$ denote the hyperplane bundle. For $k \neq l \in \mathbb{Z}$, show that H^k is not isomorphic to H^l.
4. Let $K \to \mathbb{CP}^n$ denote the bundle of $(n,0)$-forms (called the ***canonical bundle*** of $\mathbb{CP}^n$). Show that $K \cong H^{-(n+1)}$.
5. Show that $T'\mathbb{CP}^1 \cong H^2$.
6. Let M be a complex manifold. A *holomorphic vector field* on M is a holomorphic section of $T'M$. Let Z be a smooth section of $T'M$ and let θ_t denote the flow of $\operatorname{Re} Z$. Show that Z is holomorphic if and only if θ_t is a holomorphic map (where it's defined) for each t.