Math 549 **Complex Manifolds** **Spring 2019**

Assignment #1

Due 4/24/19

1. A map $F\colon \mathbb{CP}^1 \to \mathbb{CP}^n$ is called *rational* if it is of the form

$$F([Z^0, Z^1]) = [p_0(Z^0, Z^1), \ldots, p_n(Z^0, Z^1)],$$

where $p_0, \ldots, p_n$ are polynomials of some fixed degree d whose only common zero is the origin.

   (a) Show that every rational map is holomorphic.

   (b) Let $i\colon \mathbb{C} \hookrightarrow \mathbb{CP}^1$ and $j\colon \mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ be the usual embeddings. If $F\colon \mathbb{CP}^1 \to \mathbb{CP}^n$ is a rational map whose image has nontrivial intersection with $j(\mathbb{C}^n)$, show that there is a finite set $S \subseteq \mathbb{C}$ such that $j^{-1} \circ F \circ i$ maps $\mathbb{C} \smallsetminus S$ to $\mathbb{C}^n$ and has the form

$$j^{-1} \circ F \circ i(z) = (r_1(z), \ldots, r_n(z)),$$

   where $r_1, \ldots, r_n$ are rational functions (i.e., quotients of polynomials). [Remark: This explains the reason for the word "rational."]

   (c) Show that every holomorphic map from $\mathbb{CP}^1$ to itself is rational. [Hint: Show that it suffices to consider maps that fix the point at infinity.]

2. A smooth algebraic curve in $\mathbb{CP}^n$ is called *rational* if it is the image of a rational embedding $F\colon \mathbb{CP}^1 \to \mathbb{CP}^n$. Show that every nondegenerate quadric curve in $\mathbb{CP}^2$ is rational. [Hint: Consider the affine curve $xy = 1$.]

3. An *automorphism* of a complex manifold M is a biholomorphism $f\colon M \to M$.

   (a) Show that every automorphism of $\mathbb{C}$ is an affine function of the form $f(z) = az+b$ for some $a, b \in \mathbb{C}$.

   (b) Show that every automorphism of $\mathbb{CP}^1$ is a projective transformation. [Remark: We'll prove later that this is true for $\mathbb{CP}^n$ as well, but the proof requires more machinery.]

4. For any two vectors $v, w \in \mathbb{C}$ that are linearly independent over $\mathbb{R}$, let $T_{v,w} = \mathbb{C}/\langle v, w\rangle$ denote the 1-dimensional complex torus obtained as a quotient of $\mathbb{C}$ by the group of translations generated by v and w.

   (a) For any such v, w, show that there exists $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ such that $T_{v,w}$ is biholomorphic to $T_{1,\tau}$.

   (b) Let $\mathrm{SL}(2,\mathbb{Z})$ denote the group of integer matrices with determinant 1. If $\operatorname{Im}\tau > 0$ and $\operatorname{Im}\tau' > 0$, show that $T_{1,\tau}$ is biholomorphic to $T_{1,\tau'}$ if and only if there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ such that $\tau' = (a\tau + b)/(c\tau + d)$. [Hint: Show that any biholomorphism $T_{1,\tau} \to T_{1,\tau'}$ lifts to an automorphism of $\mathbb{C}$.]

5. Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $F\colon M \to N$ is said to be *conformal* if $F^*h = \lambda g$ for some smooth, positive function λ on M.

   (a) Let (M, g) and (N, h) be oriented Riemannian 2-manifolds, and give M and N the complex structures determined by their metrics and orientations. Suppose $F\colon M \to N$ is a local diffeomorphism. Show that F is holomorphic if and only if it is conformal and orientation-preserving.

   (b) Give examples of diffeomorphisms $F, G\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that F is holomorphic but not conformal, and G is conformal and orientation-preserving but not holomorphic.

6. Let M be a smooth manifold and let J be an almost complex structure on M. Define a map $N\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

   (a) Show that N is bilinear over $C^\infty(M)$, and therefore defines a $(1, 2)$-tensor field on M, called the ***Nijenhuis tensor of J***.

   (b) Show that J is integrable if and only if $N \equiv 0$. [Hint: Take X and Y to be smooth sections of $T'M$ or $T''M$.]

7. AN ALMOST COMPLEX STRUCTURE ON $\mathbb{S}^6$: Let $\mathbb{O}$ denote the algebra of *octonions* (see Problem 8-7 in *Introduction to Smooth Manifolds*). For $P, Q \in \mathbb{O}$, define $P^* = (p^*, -q)$ where $P = (p, q) \in \mathbb{O} = \mathbb{H} \times \mathbb{H}$. Let $\mathbb{R} = \{P \in \mathbb{O} : P^* = P\}$ denote the set of real octonions, identified with the real numbers in the natural way, and $\mathbb{E} = \{P \in \mathbb{O} : P^* = -P\}$ the set of imaginary octonions. We can define an inner product on $\mathbb{O}$ by $\langle P, Q\rangle = \frac{1}{2}(P^*Q + Q^*P) \in \mathbb{R}$. Let $\mathbb{S} = \{P \in \mathbb{E} : |P| = 1\}$ be the unit sphere in $\mathbb{E}$, and for each $P \in \mathbb{S}$, define a map $J_P\colon T_P\mathbb{S} \to \mathbb{O}$ by $J_P(Q) = QP$, where we identify $T_P\mathbb{S}$ with the subspace $P^\perp \cap \mathbb{E} \subseteq \mathbb{O}$.

   (a) Show that J_P maps $T_P\mathbb{S}$ to itself, and defines an almost complex structure on $\mathbb{S}$.

   (b) Show that this almost complex structure is not integrable.

   [Remark: It is still unknown whether $\mathbb{S}^6$ admits an integrable almost complex structure. Many well-known and respected mathematicians have written papers purporting to answer this question one way or the other, but all the proofs have been found to be wrong or incomplete.]