1. Let \( E \to M \) be a smooth complex vector bundle, and let \( \overline{E} \) be the complex vector bundle whose fiber \( \overline{E}_x \) at each point \( x \in M \) is equal to \( E_x \), but with complex multiplication defined by \( (a, v) \mapsto \bar{a}v \). Show that \( \overline{E} \) is isomorphic to \( E^* \) but not necessarily to \( E \).

2. Let \( M \) be a complex manifold, and let \( \pi : E \to M \) be a smooth complex vector bundle. A Cauchy-Riemann operator on \( E \) is a \( \mathbb{C} \)-linear map \( \overline{\partial} : \Gamma(E) \to \Gamma(\Lambda^{0,1} \otimes E) \) satisfying

   (i) \( \overline{\partial}(f \sigma) = (\overline{\partial} f) \otimes \sigma + f \overline{\partial} \sigma \) for all smooth complex-valued functions \( f \).

   (ii) \( \overline{\partial} (W \sigma) - W (\overline{\partial} \sigma) = [\overline{\partial}, W] \sigma \) for all \( \overline{\partial}, W \in T''M \).

   (In part (ii), we define \( \overline{\partial} \sigma \) as in Problem 8 on Assignment 3. It follows from that problem that every holomorphic vector bundle admits a Cauchy-Riemann operator.) If \( E \) is endowed with a Cauchy-Riemann operator, show that \( E \) has a unique structure as a holomorphic vector bundle such that the holomorphic sections of \( E \) are exactly those in the kernel of \( \overline{\partial} \). [Hint: If \( (e_k) \) is a smooth local frame for \( E \) over \( U \subset M \), show that the \( (0,1) \)-forms \( \theta^i_k \) on \( U \) defined by \( \overline{\partial} e_k = \theta^i_k \otimes e_j \) satisfy \( \overline{\partial} \theta^i_k + \theta^j_i \wedge \theta^k_i = 0 \). Let \( (z^j) \) be local holomorphic coordinates for \( U \) and let \( (z^j, b^k) \) be the (complex-valued) coordinates on \( \pi^{-1}(U) \subset E \) defined by the local frame \( (e_k) \), via the correspondence \( (z^j, b^k) \leftrightarrow b^k e_k \mid z \). Show that there is a unique integrable complex structure on the total space of \( E \) such that \( \Lambda^{1,0}E \) is locally spanned by \( (\pi^* dz^j, db^j + b^k \pi^* \theta^j_k) \), and apply the Newlander-Nirenberg theorem.]

3. Let \( \Sigma \) be a Riemann surface and let \( g \) be a Kähler metric on \( \Sigma \). If \( z \) is any local holomorphic coordinate on \( \Sigma \), show that the holomorphic sectional curvature of \( g \) is equal to its Gaussian curvature, and both are equal to

   \[-\frac{1}{u} \frac{\partial^2}{\partial z \partial \overline{z}} \log u,\]

   where \( u = g_C(\partial/\partial z, \partial/\partial \overline{z}) \). Use this formula to compute the Gaussian curvatures of the 1-dimensional Fubini-Study and complex hyperbolic metrics.

4. Let \( Q \subset \mathbb{CP}^2 \) be the quadric curve defined by the homogeneous polynomial \( z^1 z^2 - (z^3)^2 \). Compute the Gaussian curvature and the area of \( Q \) in the metric obtained by restricting the Fubini-Study metric to \( Q \).

5. Let \( E \to M \) be a smooth complex vector bundle of rank \( k \). Show that \( c^R_1(E) = c^R_1(\Lambda^k E) \), where \( \Lambda^k E \) denotes the bundle of antisymmetric contravariant \( k \)-tensors on \( E \) and \( c^R_1 \) denotes the real first Chern class.
6. **PROBLEM DELETED**

7. Let \( \pi: E \to M \) and \( \pi': E' \to M' \) be smooth complex vector bundles of rank \( k \), and let \( F: E \to E' \) be a smooth bundle map covering \( f: M \to M' \). (Recall that this means \( \pi' \circ F = f \circ \pi \), and for each \( x \in M \), the map \( F_x = F|_{E_x}: E_x \to E'_x \) is a linear isomorphism.)

(a) If \( (e'_j) \) is a smooth frame for \( E' \) over an open set \( U' \subset M' \), show that there is a unique smooth frame \( (e_j) \) for \( E \) over \( f^{-1}(U') \) such that \( F \circ e_j = e'_j \circ f \) for each \( j \).

(b) If \( \nabla' \) is a connection on \( E' \), show that there is a unique connection \( \nabla \) on \( E \), called the pullback connection, with the property that

\[
\nabla_X e_j = F^{-1}_x \nabla'_{f^*X}(e'_j)
\]

whenever the frames \( (e_j) \) and \( (e'_j) \) are related as in part (a).

(c) For each \( j = 1, \ldots, k \), show that \( c^R_j(E) = f^* c^R_j(E') \).

8. (a) Show that \( U(n+1) \) acts transitively on \( \mathbb{CP}^n \) by projective transformations.

(b) Show that the Fubini-Study metric is \( U(n+1) \)-invariant, and is, up to a constant multiple, the unique \( U(n+1) \)-invariant metric on \( \mathbb{CP}^n \).

9. (a) Let \( U(n, 1) \) be the subgroup of \( GL(n+1, \mathbb{C}) \) leaving invariant the following hermitian bilinear form:

\[
H = dz^1 \otimes \overline{dz^1} + \cdots + dz^n \otimes \overline{dz^n} - dz^{n+1} \otimes \overline{dz^{n+1}}.
\]

Considering the unit ball \( \mathbb{B}^{2n} \subset \mathbb{C}^n \subset \mathbb{CP}^n \) as a subset of projective space, show that \( U(n, 1) \) acts transitively on \( \mathbb{B}^{2n} \) by projective transformations.

(b) Let \( g \) be the complex hyperbolic metric on \( \mathbb{B}^{2n} \), defined by the Kähler form \( \omega = \frac{i}{2} \partial \overline{\partial} \log(|z|^2 - 1) \). Show that \( g \) is, up to a constant multiple, the unique \( U(n, 1) \)-invariant metric on \( \mathbb{B}^{2n} \).

(c) Show that \( g \) has constant holomorphic sectional curvature equal to \(-4\).

10. Let \( M \) be a complex manifold of dimension \( n \), and let \( g \) be a Kähler metric on \( M \) with constant holomorphic sectional curvature \( C \).

(a) Let \( X, Y \in T_x M \) be a pair of orthonormal vectors. Show that the (ordinary) sectional curvature of \( g \) in the direction of the plane spanned by \( (X, Y) \) is given by

\[
K(X, Y) = \frac{1}{4} C \left( 1 + 3 \langle X, JY \rangle^2 \right).
\]

(b) If \( n \geq 2 \), show that at each point of \( M \), the (ordinary) sectional curvatures of \( g \) take on all values between \( \frac{1}{4} C \) and \( C \), inclusive.