Math 549 **Complex Manifolds** **Spring 2004**

ASSIGNMENT #3

DUE 5/24/04

1. If $\mathscr{S} \to M$ is a sheaf, $U \subset M$ is open, and $\sigma, \tau \in \mathscr{S}(U)$ are sections, show that the set $\{x \in U : \sigma(x) = \tau(x)\}$ is open. If U is connected and $\sigma = \tau$ somewhere, does this imply that $\sigma \equiv \tau$ on U?

2. Let M be a topological manifold and let $\mathscr{S}$, $\mathscr{T}$ be sheaves over M. Show that every sheaf homomorphism $F\colon \mathscr{S} \to \mathscr{T}$ is a local homeomorphism.

3. Let M be a complex n-manifold, and for $0 \le q \le n$ let Ω^q denote the sheaf of holomorphic q-forms, i.e., $\overline{\partial}$-closed $(q,0)$-forms. (Thus sections of Ω^0 are just holomorphic functions.) For every holomorphic map $f\colon M \to N$, show that there is a group homomorphism $f^*\colon H^p(N;\Omega^q) \to H^p(M;\Omega^q)$, such that $\mathrm{Id}^* = \mathrm{Id}$ and $(f \circ g)^* = g^* \circ f^*$. Conclude that $H^p(M;\Omega^q)$ is a biholomorphism invariant. Give a counterexample to show that it need not be a diffeomorphism invariant.

4. Suppose $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ are smooth vector bundles and $F\colon E \to E'$ is a smooth bundle map covering $f\colon M \to M'$. (This means that f and F are smooth, and for each $x \in M$, F restricts to a linear isomorphism from E_x to $E'_{f(x)}$.) Show that $f^* c_l(E') = c_1(E)$.

5. Let M be a smooth manifold, and let $H^k(M;\underline{\mathbb{R}})$ denote sheaf cohomology with coefficients in the constant sheaf $\underline{\mathbb{R}}$. Let $\mathscr{U} = \{U_\alpha\}$ be an open covering of M such that each nonempty finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ is contractible. By threading through the proof of the generalized de Rham theorem, show that the de Rham isomorphisms $\mathscr{I}_1\colon H^1_{\mathrm{dR}}(M) \to H^1(M;\underline{\mathbb{R}})$ and $\mathscr{I}_2\colon H^2_{\mathrm{dR}}(M) \to H^2(M;\underline{\mathbb{R}})$ can be described as follows.

   (a) Let η be a closed 1-form on M. For each α, there is a smooth function u_α on U_α such that $\eta|_{U_\alpha} = du_\alpha$. Then $a(U_\alpha, U_\beta) = u_\beta|_{U_\alpha \cap U_\beta} - u_\alpha|_{U_\alpha \cap U_\beta}$ defines a 1-cocycle on $\mathscr{U}$ with coefficients in $\underline{\mathbb{R}}$, and $\mathscr{I}_1[\eta] = [a]$.

   (b) Let η be a closed 2-form on M. For each α, there is a smooth 1-form φ_α on U_α such that $\eta|_{U_\alpha} = d\varphi_\alpha$; and for each α and β such that $U_\alpha \cap U_\beta \ne \emptyset$, there is a smooth function $u_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that $\varphi_\beta|_{U_\alpha \cap U_\beta} - \varphi_\alpha|_{U_\alpha \cap U_\beta} = du_{\alpha\beta}$. Then $a(U_\alpha, U_\beta, U_\gamma) = \left(u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha}\right)\big|_{U_\alpha \cap U_\beta \cap U_\gamma}$ defines a 2-cocycle on $\mathscr{U}$ with coefficients in $\underline{\mathbb{R}}$, and $\mathscr{I}_2[\eta] = [a]$.

6. Let M be a complex manifold. A smooth, real-valued function u on M is said to be *pluriharmonic* if in any holomorphic coordinates, u is harmonic (in the usual Euclidean sense) as a function of each complex coordinate when the others are held fixed. Show that the following are equivalent.

   (a) u is pluriharmonic.

   (b) $\partial\overline{\partial} u = 0$.

(c) For every holomorphic embedding $j\colon D \hookrightarrow M$ of the unit disk D into M, j^*u is harmonic (in the usual Euclidean sense) on D.

(d) In a neighborhood of each point, u is the real part of a holomorphic function.

7. Let M be a complex manifold, and let $\mathscr{P}$ denote the sheaf of (germs of) pluriharmonic functions on M. For each $q \geq 1$, let $\mathscr{F}^q$ denote the sheaf of real $(q+1)$-forms whose $(q+1,0)$ and $(0,q+1)$-parts are zero; in other words, $\mathscr{F}^q$ is the sheaf of real-valued forms in $\mathscr{E}^{(q,1)} \oplus \cdots \oplus \mathscr{E}^{(1,q)}$. Show that the following sheaf sequence is exact:

$$0 \longrightarrow \mathscr{P} \hookrightarrow \mathscr{E}^0 \xrightarrow{i\partial\overline{\partial}} \mathscr{F}^1 \xrightarrow{d} \mathscr{F}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{F}^q \xrightarrow{d} \cdots .$$

Conclude that for $q \geq 2$, $H^q(M;\mathscr{P})$ is isomorphic to the kernel of $d\colon \mathscr{F}^q(M) \to \mathscr{F}^{q+1}(M)$ modulo the image of $d\colon \mathscr{F}^{q-1}(M) \to \mathscr{F}^q(M)$. State the analogous result for $q = 1$. [Hint: For the proof of exactness at $\mathscr{F}^q$, if β is a local section of $\mathscr{F}^q$ and $\beta = d\alpha$, write $\alpha = \alpha^{(q,0)} + \widetilde{\alpha} + \alpha^{(0,q)}$ with $\widetilde{\alpha}$ a section of $\mathscr{F}^{q-1}$, and show that locally $d\alpha^{(q,0)} = d\overline{\partial}\sigma$ for some $(q-1,0)$ form σ.]

8. Let M be a complex manifold, and let E be a holomorphic vector bundle over M.

   (a) Show that the operator $\overline{\partial}\colon \Gamma(E) \to \Gamma(\Lambda^{0,1}M \otimes E)$ defined in class satisfies the following two properties.

      i. $\overline{\partial}(f\sigma) = (\overline{\partial}f) \otimes \sigma + f\overline{\partial}\sigma$.

      ii. $\overline{Z}(\overline{W}\sigma) - \overline{W}(\overline{Z}\sigma) = [\overline{Z},\overline{W}]\sigma$, where $\overline{Z}\sigma$ is shorthand for the section of E obtained by inserting $\overline{Z}$ into the $\Lambda^{0,1}$ slot of $\overline{\partial}\sigma$.

   (b) Show that $\overline{\partial}$ extends to an operator $\overline{\partial}\colon \Gamma(\Lambda^{p,q} \otimes E) \to \Gamma(\Lambda^{p,q+1}M \otimes E)$ satisfying

   $$\overline{\partial}(\alpha \otimes \sigma) = \overline{\partial}\alpha \otimes \sigma + (-1)^{p+q}\alpha \wedge \overline{\partial}\sigma,$$

   where α is a smooth (p,q)-form, σ is a smooth section of E, and the wedge product is between the differential form components of α and $\overline{\partial}\sigma$.

   (c) Show that $\overline{\partial} \circ \overline{\partial} = 0$.

9. Let $E \to M$ be a smooth complex vector bundle, and let ∇ be any connection on E. For any $x \in M$, show that there exists a smooth local frame (e_j) for E in a neighborhood of x such that $\nabla e_j = 0$ at x.

10. **Optional:** *(This exercise is aimed primarily at those who know something about simplicial complexes and simplicial cohomology; see, for example, Chapters 5 and 13 of [ITM] and Chapter 5 of Munkres's Elements of Algebraic Topology.)* Let K be an abstract simplicial complex, and let $|K|$ denote the underlying topological space of K. Let $\mathscr{U}$ be the open cover of $|K|$ defined by $\mathscr{U} = \{\operatorname{St} v : v \in K^{(0)}\}$, where $K^{(0)}$ is the set of vertices of K and $\operatorname{St} v$ is the open star of v [ITM, Problem 5-2]. If G is an abelian group, for each nonnegative integer k, define a *simplicial k-cochain* in K with coefficients in G to be a function c from the set of ordered k-simplices in K to G such that c changes sign whenever two vertices are interchanged:

$$c(v_0,\dots,v_i,\dots,v_j,\dots,v_k) = -c(v_0,\dots,v_j,\dots,v_i,\dots,v_k).$$

Let $C^k_\Delta(K;G)$ denote the set of all such k-cochains, which is an abelian group under the obvious operation of addition of cochains. Define coboundary operators $\delta_k\colon C^k_\Delta(K;G) \to C^{k+1}_\Delta(K;G)$ by

$$(\delta_k c)(v_0, \dots, v_{k+1}) = \sum_{j=0}^{k+1} (-1)^j c(v_0, \dots, \widehat{v}_j, \dots, v_{k+1}),$$

and let $H^k_\Delta(K;G) = \operatorname{Ker} \delta_k / \operatorname{Im} \delta_{k-1}$ denote the resulting cohomology groups. Let $C^k(\mathscr{U};\underline{G})$ denote the group of k-cochains on $\mathscr{U}$ with coefficients in the constant sheaf $\underline{G}$, and define a group homomorphism $\Phi\colon C^k(\mathscr{U};\underline{G}) \to C^k_\Delta(K;G)$ by

$$\Phi(c)(v_0, \dots, v_k) = c(\operatorname{St} v_0, \dots, \operatorname{St} v_k).$$

Show that Φ descends to an isomorphism $H^k(|K|;\underline{G}) \to H^k_\Delta(K;G)$. You may use without proof either or both of the following theorems:

Subdivision Theorem: If $\mathscr{U}$ is any open cover of $|K|$, there is a subdivision $\widetilde{K}$ of K such that the covering $\{\operatorname{St} v : v \in \widetilde{K}^{(0)}\}$ refines $\mathscr{U}$, and a simplicial map $F\colon \widetilde{K} \to K$ (called a *simplicial approximation to the identity*) such that the induced map $F^*\colon H^k_\Delta(K;G) \to H^k_\Delta(\widetilde{K};G)$ is an isomorphism. (See Munkres, *Elements of Algebraic Topology*, for a proof.)

Leray's Theorem: Suppose $\mathscr{S} \to M$ is a sheaf over a paracompact space M, and $\mathscr{U}$ is an open cover of M with the property that the restriction of $\mathscr{S}$ to $U_0 \cap \cdots \cap U_k$ is acyclic for all finite collections $\{U_0, \dots, U_k\} \subset \mathscr{U}$ with nonempty intersection. Then the natural map $H^k(\mathscr{U};\mathscr{S}) \to H^k(M;\mathscr{S})$ is an isomorphism. (See [G] for a proof.)