Math 549 **Complex Manifolds** **Spring 2004**

ASSIGNMENT #1

DUE 4/26/04

1. Show that every complex manifold has a canonical orientation, uniqueley determined by the following two properties:
   (a) The frame $(\partial/\partial x^1, \partial/\partial y^1, \dots, \partial/\partial x^n, \partial/\partial y^n)$ is oriented for $\mathbb{C}^n$ with its standard complex structure.
   (b) Every local biholomorphism is orientation-preserving.
2. Let $U \subset \mathbb{C}^n$ be an open set, and let $F\colon U \to \mathbb{C}^m$ be a smooth map. Show that F is holomorphic if and only if $F_* \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ F_*$.
3. Let (M, J) be an almost complex manifold, and define $\Lambda^{p,q}M$ just as we did for complex manifolds. Show that the following are equivalent:
   (a) J is integrable.
   (b) For each pair of nonnegative integers p, q, the exterior derivative d maps sections of $\Lambda^{p,q}M$ to sections of $\Lambda^{p+1,q}M \oplus \Lambda^{p,q+1}M$.
   (c) d maps sections of $\Lambda^{0,1}M$ to sections of $\Lambda^{1,1}M \oplus \Lambda^{0,2}M$.
4. Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $F\colon M \to N$ is said to be *conformal* if $F^*h = \lambda g$ for some smooth, positive function λ on M.
   (a) Let (M, g) and (N, h) be oriented Riemannian 2-manifolds, and give M and N the complex structures determined by their metrics and orientations. Suppose $F\colon M \to N$ is a local diffeomorphism. Show that F is holomorphic if and only if it is conformal and orientation-preserving.
   (b) Give examples of diffeomorphisms $F, G\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that F is holomorphic but not conformal, and G is conformal and orientation-preserving but not holomorphic.
5. A map $F\colon \mathbb{CP}^1 \to \mathbb{CP}^n$ is called *rational* if it is of the form
$$F([z^1, z^2]) = [p_1(z^1, z^2), \dots, p_{n+1}(z^1, z^2)],$$
where $p_1, \dots, p_{n+1}$ are polynomials of some fixed degree d whose only common zero is the origin.
   (a) Show that every rational map is holomorphic.
   (b) Let $i\colon \mathbb{C} \hookrightarrow \mathbb{CP}^1$ and $j\colon \mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ be the usual embeddings. If $F\colon \mathbb{CP}^1 \to \mathbb{CP}^n$ is a rational map whose image has nontrivial intersection with $j(\mathbb{C}^n)$, show that there is a finite set $S \subset \mathbb{C}$ such that $j^{-1} \circ F \circ i$ maps $\mathbb{C} \setminus S$ to $\mathbb{C}^n$ and has the form
$$j^{-1} \circ F \circ i(z) = (r_1(z), \dots, r_n(z)),$$
where $r_1, \dots, r_n$ are rational functions (i.e., quotients of polynomials).

(c) Show that every holomorphic map from $\mathbb{CP}^1$ to itself is rational. [Hint: show that it suffices to consider maps that fix the point at infinity.]

6. A smooth algebraic curve in $\mathbb{CP}^n$ is called *rational* if it is the image of a rational embedding $F\colon \mathbb{CP}^1 \to \mathbb{CP}^n$. Show that every nondegenerate quadric curve in $\mathbb{CP}^2$ is rational. [Hint: Consider the affine curve $xy = 1$.]

7. An *automorphism* of a complex manifold M is a biholomorphism $f\colon M \to M$.

   (a) Show that every automorphism of $\mathbb{C}$ is an affine function of the form $f(z) = az+b$ for some $a, b \in \mathbb{C}$.

   (b) Show that every automorphism of $\mathbb{CP}^1$ is a projective transformation.

8. For any two vectors $v, w \in \mathbb{C}$ that are linearly independent over $\mathbb{R}$, let $T_{v,w} = \mathbb{C}/\langle v, w\rangle$ denote the 1-dimensional complex manifold obtained as a quotient of $\mathbb{C}$ by the group of translations generated by v and w.

   (a) For any such v, w, show that there exists $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ such that $T_{v,w}$ is biholomorphic to $T_{1,\tau}$.

   (b) Let $\mathrm{SL}(2,\mathbb{Z})$ denote the group of integer matrices with determinant 1. If $\operatorname{Im}\tau > 0$ and $\operatorname{Im}\tau' > 0$, show that $T_{1,\tau}$ is biholomorphic to $T_{1,\tau'}$ if and only if there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ such that $\tau' = (a\tau + b)/(c\tau + d)$. [Hint: show that any biholomorphism $T_{1,\tau} \to T_{1,\tau'}$ lifts to an automorphism of $\mathbb{C}$.]