Math 548 **Geometric Structures** **Winter 1997**

Assignment #4

Due 3/21/97

1. Let $P \to M$ be a principal G-bundle, and $\theta\colon G \times F \to F$ a smooth, effective action. Define the associated F-bundle $E \to M$ by

$$E = P \times_\theta F := P \times F / \sim,$$

where $(p, f) \sim (p \cdot g, g^{-1} \cdot f)$ for $g \in G$. Prove that E has a natural structure as a fiber bundle with fiber F and structure group G, whose associated principal bundle is isomorphic to P.

2. Using Problem 4(b) of Assignment 3, together with the fact that $SO(n+1)$ acts transitively on $\mathbf{S}^n$ with isotropy group $SO(n)$, show that $SO(n+1)$ is the total space of a principal $SO(n)$ bundle over $\mathbf{S}^n$. Show that this bundle is isomorphic to $SO(\mathbf{S}^n)$, the bundle of oriented orthonormal frames on $\mathbf{S}^n$.

3. (a) Show that the map $f\colon S^1 \to \mathbf{RP}^2$ given by

$$f(e^{i\theta}) = [\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, 0]$$

is a classifying map for the Möbius bundle.

(b) Let $U \to \mathbf{CP}^1$ denote the universal complex line bundle over $\mathbf{CP}^1$. For $k > 0$, show that the map

$$p_k[z, w] = [z^k, w^k]$$

is a classifying map for $U^k = U \otimes \ldots \otimes U$; and that

$$p_{-k}[z, w] = [\bar{z}^k, \bar{w}^k]$$

is a classifying map for $\overline{U}^k$.

4. Let w_1 and c_1 denote the first Stiefel-Whitney and (integral) first Chern classes, respectively. Prove the addition formulas

$$c_1(E \oplus \widetilde{E}) = c_1(E) + c_1(\widetilde{E});$$
$$w_1(E \oplus \widetilde{E}) = w_1(E) + w_1(\widetilde{E}).$$

5. Let $E \to M$ be a k-dimensional (real or complex) vector bundle. Suppose there exists a global section Ω of $\Lambda_k E$, and give E the corresponding structure group $G = SL(k, \mathbf{R})$ or $SL(k, \mathbf{C})$. Prove that a connection ∇ on E is a G-connection if and only if $\nabla\Omega = 0$.

6. Let $P \to M$ be a principal G-bundle over an n-manifold M, and $H \subset TP$ an n-dimensional plane field that is everywhere transverse to the vertical plane field V. A piecewise smooth curve $\sigma\colon I \to P$ is said to be ***horizontal*** if $\dot\sigma(t) \in H_{\sigma(t)}$ for all $t \in I$. (At points where σ is not smooth, this means that both one-sided derivatives are in $H_{\sigma(t)}$.) If $\gamma\colon I \to M$ is a piecewise smooth curve in M, σ is called a ***lift*** of γ if $\pi \circ \sigma = \gamma$.

   (a) Given any curve $\gamma\colon I \to M$ and any point $p \in P_{\gamma(0)}$, show γ has a unique horizontal lift σ such that $\sigma(0) = p$.

   (b) Now suppose H is the horizontal plane field associated to a connection ω in P, E is a vector bundle associated to P, and ∇ is the G-connection in E determined by ω. Show that, if $\sigma\colon I \to P$ is any lift of $\gamma\colon I \to M$, then the frame $(\sigma_1, \ldots, \sigma_k)$ for E along γ determined by σ is parallel if and only if σ is horizontal.

7. Let M be a connected smooth manifold and $x \in M$. Suppose $E \to M$ is a flat vector bundle (i.e. a vector bundle with a flat connection ∇). For any piecewise smooth closed curve γ starting and ending at x, parallel translation around γ defines a linear map $P_\gamma\colon E_x \to E_x$.

   (a) Show that P_γ depends only on the homotopy class of γ in $\pi_1(M, x)$.

   (b) Show that the map $P\colon \pi_1(M, x) \to GL(E_x)$ so defined is a representation of $\pi_1(M, x)$, called the ***holonomy representation***.

   (c) We say that two flat bundles (E, ∇) and $(\widetilde{E}, \widetilde{\nabla})$ over M are equivalent if there is a bundle isomorphism $F\colon E \to \widetilde{E}$ such that $F^*\widetilde{\nabla} = \nabla$. For any group Γ, two representations $\rho\colon \Gamma \to GL(V)$, $\tilde\rho\colon \Gamma \to GL(\widetilde{V})$ are said to be equivalent if there is an isomorphism $\varphi\colon V \to \widetilde{V}$ such that $\varphi \circ \rho(g) = \tilde\rho(g) \circ \varphi$ for all $g \in \Gamma$. Show that the holonomy representation gives a one-to-one correspondence between equivalence classes of finite-dimensional representations of $\pi_1(M, x)$ and equivalence classes of flat bundles over M.

8. (a) Prove that the Maurer-Cartan form ξ on any Lie group satisfies the following identity, known as the ***Maurer-Cartan equation***:

$$d\xi + \xi \wedge \xi = 0.$$

   (b) If $P = M \times G \to M$ is the canonical trivial principal G-bundle over M, show that $\pi_2^*\xi$ is a flat connection on P, where π_2 is projection onto G and ξ is the Maurer-Cartan form of G.