Math 548 **Geometric Structures** **Winter 1997**

Assignment #3

Due 2/28/97

1. (a) Let $E \to \mathbf{S}^1$ be the Möbius bundle. Show that it admits a reduction to $\mathbf{Z}_2 \subset GL(1,\mathbf{R})$, and that the associated principal bundle is a double covering of $\mathbf{S}^1$ by itself.

   (b) Show that any principal $\mathbf{S}^1$ bundle over $\mathbf{S}^1$ is trivial.

   (c) Show that there is a nontrivial fiber bundle over $\mathbf{S}^1$ with fiber $\mathbf{S}^1$ and structure group $\mathbf{Z}_2$.

2. Let G be a closed Lie subgroup of $GL(n,\mathbf{R})$. Recall that a ***G-structure*** on an n-manifold M is a reduction of the structure group of TM to G. A G-structure is said to be ***integrable*** if, in a neighborhood of every point, there exist coordinates $\{x^i\}$ such that $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ is a G-frame. Show that the following pairs of structures are equivalent.

   (a) Integrable $O(n)$-structures and flat Riemannian metrics.

   (b) Integrable $GL(k,\mathbf{C})$-structures and complex structures, where $n = 2k$. [By identifying $\mathbf{R}^{2k}$ with $\mathbf{C}^k$ via the correspondence $(x^1, y^1, \dots, x^k, y^k) \leftrightarrow (x^1 + iy^1, \dots, x^k + iy^k)$, we can consider $GL(k,\mathbf{C})$ as a subgroup of $GL(2k,\mathbf{R})$. A ***complex structure*** on a $2k$-dimensional manifold M is a maximal atlas of coordinate charts taking their values in $\mathbf{C}^k = \mathbf{R}^{2k}$ which overlap holomorphically.]

   (c) Integrable $Sp(k,\mathbf{R})$-structures and symplectic structures, where $n = 2k$. [Here $Sp(k,\mathbf{R})$ is the group of linear transformations of $\mathbf{R}^{2k}$ preserving the 2-form $\sum dx^j \wedge dy^j$. Hint: for one direction, use the Darboux theorem.]

   (d) Integrable G_m structures and m-dimensional foliations, where G_m is the subgroup of $GL(n,\mathbf{R})$ consisting of matrices that map $\mathbf{R}^m$ to itself.

   (e) Integrable $SL(n,\mathbf{R})$ structures and $SL(n,\mathbf{R})$ structures.

3. Prove the following theorem stated in class. *Suppose P is a smooth manifold, and G is a Lie group acting smoothly, freely, and properly on the right on P. Then the quotient space $M := P/G$ has a canonical smooth manifold structure such that P is a principal G-bundle over M with the given right action.* [Hint: imitate the proof that the quotient of a Lie group by a closed Lie subgroup is a manifold; see Boothby or Warner for details.]

4. (a) Let $p\colon \widetilde{M} \to M$ be a normal covering space. (Recall that normality means that $p_*\pi_1(\widetilde{M})$ is normal in $\pi_1(M)$, or equivalently that the group Γ of covering transformations acts transitively on fibers.) Use the result of Problem 3 to show that $\widetilde{M}$ is a principal Γ-bundle over M with the right action $R\colon \widetilde{M} \times \Gamma \to \widetilde{M}$ given by $R(x,\gamma) = \gamma^{-1}(x)$.

(b) Let G be a Lie group and H a closed Lie subgroup. Use the result of Problem 3 to show that the set G/H of left cosets is a smooth manifold admitting a smooth, transitive (left) G-action.

5. (a) Complete the proof that $Spin(4) \cong SU(2) \times SU(2)$.

(b) Let $\sigma\colon Spin(4) \to GL(W_+) \times GL(W_-) \subset GL(W)$ denote the spin representation; $\rho\colon Spin(4) \to GL(V)$ the orthogonal representation; $\rho_{\mathbf{C}}$ the complexification of ρ; and $\tau\colon Spin(4) \to GL(\mathrm{End}(W_+, W_-))$ the representation defined by

$$(\tau(p,q)A)\psi_+ = \sigma(p)A\sigma(q)^{-1}\psi_+.$$

Show that τ and $\rho_{\mathbf{C}}$ are isomorphic representations.