Math 548 **Geometric Structures** **Winter 1997**

ASSIGNMENT #2

DUE 2/14/97

1. Let V be a finite-dimensional vector space, $G_k(V)$ the Grassmann manifold of k-dimensional subspaces of V, and $U_k(V)$ the subset of $G_k(V) \times V$ defined by $U_k(V) := \{(\xi, x) : x \in \xi\}$. Carry out the complete proof that $U_k(V)$ is a smooth k-dimensional vector bundle over $G_k(V)$ with the obvious projection.

2. (VECTOR BUNDLE CONSTRUCTION THEOREM) Suppose we are given a covering of a smooth manifold M by open sets $\mathfrak{U}_\alpha$, and, for each pair $\mathfrak{U}_\alpha, \mathfrak{U}_\beta$ with nonempty intersection, a smooth map $\tau_{\alpha\beta} \colon \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta \to GL(k)$ (where $GL(k)$ can denote either $GL(k, \mathbf{R})$ or $GL(k, \mathbf{C})$) such that the following ***cocycle conditions*** are satisfied:

$$\tau_{\beta\alpha}(x) = \tau_{\alpha\beta}(x)^{-1} \text{ for all } x \in \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta;$$
$$\tau_{\alpha\beta}(x)\tau_{\beta\gamma}(x) = \tau_{\alpha\gamma}(x) \text{ for all } x \in \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta \cap \mathfrak{U}_\gamma.$$

   (a) Show that there is a k-dimensional (real or complex) vector bundle $E \to M$, such that each set $\mathfrak{U}_\alpha$ has a local trivialization $\varphi_\alpha \colon \mathfrak{U}_\alpha \times \mathbf{R}^k \to \pi^{-1}(\mathfrak{U}_\alpha)$ (or $\mathfrak{U}_\alpha \times \mathbf{C}^k \to \pi^{-1}(\mathfrak{U}_\alpha)$) whose transition functions are $\tau_{\alpha\beta}$. [Hint: define E as a suitable quotient space of $\coprod_\alpha \mathfrak{U}_\alpha \times \mathbf{R}^k$ or $\coprod_\alpha \mathfrak{U}_\alpha \times \mathbf{C}^k$.]

   (b) Suppose we are given two sets of transition functions $\{\tau_{\alpha\beta}\}$, $\{\tilde{\tau}_{\alpha\beta}\}$ defined on the same collection of open sets $\{\mathfrak{U}_\alpha\}$, and satisfying the cocycle conditions. Show that they determine isomorphic bundles if and only if there exist smooth maps $\lambda_\alpha \colon \mathfrak{U}_\alpha \to GL(k)$ such that

$$\tilde{\tau}_{\alpha\beta}(x) = \lambda_\alpha(x)^{-1}\tau_{\alpha\beta}(x)\lambda_\beta(x), \qquad \text{for all } x \in \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta.$$

3. Suppose $E \to M$ is a Euclidean or Hermitian vector bundle (i.e. a real or complex vector bundle with a fiber metric), and $S \subset E$ is a vector subbundle (i.e. a subset which is a smooth vector bundle over M with the same projection).

   (a) Show that $S^\perp$ is a vector subbundle of E.

   (b) Show that $E \cong S \oplus S^\perp$.

4. (a) Suppose $E_1 \to M_1$, $E_2 \to M_2$ are vector bundles, and $F \colon E_1 \to E_2$ is a vector bundle homomorphism with constant rank. Show that $\operatorname{Ker} F \subset E_1$ and $\operatorname{Im} F \subset E_2$ are vector subbundles.

   (b) If $E \to M$ is an m-dimensional vector bundle and $S \subset E$ a k-dimensional vector subbundle, define the ***quotient bundle*** E/S as the set

$$E/S := \coprod_{p \in M} E_p/S_p.$$

   Show that E/S has a unique structure as an $(m-k)$-vector bundle such that the natural projection $E \to E/S$ is a vector bundle homomorphism with kernel S.

(c) If E is a Euclidean or Hermitian vector bundle and $S \subset E$ is a vector subbundle, show that E/S is canonically isomorphic to $S^\perp$.

(d) Let $\widetilde{M}$ be a Riemannian manifold and $M \subset \widetilde{M}$ an (immersed or embedded) submanifold. Show that the ***algebraic normal bundle*** $NM := (T\widetilde{M}|_M)/TM$ and the ***geometric normal bundle*** $N_gM := TM^\perp \subset T\widetilde{M}|_M$ are canonically isomorphic.

5. A ***line bundle*** is a 1-dimensional vector bundle. For M a smooth manifold, let $V^1_{\mathbf{R}}(M)$ denote the set of isomorphism classes of real line bundles over M, and $V^1_{\mathbf{C}}(M)$ the set of isomorphism classes of complex line bundles.

   (a) Show that $V^1_{\mathbf{R}}(M)$ and $V^1_{\mathbf{C}}(M)$ are abelian groups under tensor product. What is the identity? What is the inverse of a bundle E?

   (b) Show every element of $V^1_{\mathbf{R}}(M)$ has order at most 2.

   (c) Compute $V^1_{\mathbf{R}}(\mathbf{S}^1)$, giving an explicit representative for each equivalence class.

6. (a) Let $E \to M$ be a real or complex k-dimensional vector bundle. Let $GL(k)$ denote either $GL(k, \mathbf{R})$ or $GL(k, \mathbf{C})$, let $\rho\colon GL(k) \to GL(k)$ be the contragredient representation $\rho(A) = (A^{-1})^t$, and let E^* be the bundle constructed from E via this representation. Show that the fibers of E^* can be naturally identified with the dual spaces of the fibers of E, i.e., for each $x \in M$ there is an $\mathbf{R}$- or $\mathbf{C}$-linear isomorphism $(E^*)_x \cong (E_x)^*$ which is independent of basis or local trivialization.

   (b) Now let E be complex, and let $\rho\colon GL(k, \mathbf{C}) \to GL(k, \mathbf{C})$ be the conjugate representation $\rho(A) = \overline{A}$. Show that each fiber $\overline{E}_x$ of the resulting bundle can be naturally identified with the conjugate of E_x, that is, the vector space which is equal to E_x as a set but whose scalar multiplication $\mathbf{C} \times E_x \to E_x$ is defined by $(c, v) \mapsto \overline{c}v$.

7. Let M be an oriented 2-manifold with a Riemannian metric. Show that there is a natural complex line bundle $T'M \to M$ whose underlying real 2-plane bundle is isomorphic to TM. [Hint: Cover M by open sets carrying oriented orthonormal frames, and construct transition functions using a representation of $SO(2)$ in $GL(1, \mathbf{C})$.]

8. Let $\mathbf{CP}^1$ denote the 1-dimensional complex projective space and $\mathbf{S}^2 \subset \mathbf{R}^3$ the unit 2-sphere.

   (a) Show that the formula $f(x, y, z) = [x + iy, 1 - z] = [1 + z, x - iy]$ gives a diffeomorphism $f\colon \mathbf{S}^2 \to \mathbf{CP}^1$.

   (b) Using the two charts $U_j := \{[w^1, w^2] \colon w^j \neq 0\}$ on $\mathbf{CP}^2$, with the usual coordinates $(x, y) \leftrightarrow [1, x + iy]$ on U_1 and $(x, y) \leftrightarrow [x + iy, 1]$ on U_2, compute the transition functions for the universal bundle $\mathbf{U} \to \mathbf{CP}^1$ and the complex line bundle $T'\mathbf{S}^2 \to \mathbf{S}^2$ constructed in Problem 7.

   (c) Prove that $T\mathbf{S}^2 \cong \mathbf{U}^* \otimes \mathbf{U}^*$.

   (d) Prove that $\mathbf{U} \not\cong \mathbf{U}^*$. [Hint: use Problem 5 and the fact that $T\mathbf{S}^2$ does not admit a global nonvanishing section.]