Math 548 **Geometric Structures** **Winter 1997**

Assignment #1

Due 1/31/97

1. Let $\Delta \subset \mathbf{R}^N$ be a geometric k-simplex, and $\mathring{\Delta}$ the open simplex with the same vertices as Δ. Observe that the union $\mathcal{D}$ of $\mathring{\Delta}$ with its open $(k-1)$-dimensional faces is a manifold-with-boundary. Show that there is a $1-1$ correspondence between orientations of Δ as a simplex (i.e. equivalence classes of orderings of its vertices) and orientations of $\mathring{\Delta}$ as a smooth k-manifold, such that the $(k-1)$-chain $\partial\Delta$ consists of the sum of the $(k-1)$-faces of Δ with their Stokes orientations as components of the boundary of $\mathcal{D}$.

2. Let K be any simplicial complex. Show that for each k there is a canonical isomorphism $\varphi\colon H^k_\Delta(K;\mathbf{R}) \to (H_k(K;\mathbf{R}))^*$ such that, for any PL map $f\colon K \to L$, the following diagram commutes:

$$\begin{array}{ccc} H^k_\Delta(L;\mathbf{R}) & \xrightarrow{\varphi} & (H_k(L;\mathbf{R}))^* \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f_*)^*} \\ H^k_\Delta(K;\mathbf{R}) & \xrightarrow{\varphi} & (H_k(K;\mathbf{R}))^* \end{array}$$

3. Complete the proof that the sequence of sheaf homomorphisms

$$0 \to \mathcal{R} \xrightarrow{i} \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \to \dots$$

is exact by proving exactness at $\mathcal{R}$ and $\mathcal{E}^0$.

4. Let $\mathcal{F} \to M$ be a sheaf, and $\mathfrak{U}, \mathcal{V}$ open covers of M such that $\mathcal{V}$ refines $\mathfrak{U}$. For any refining map $\rho\colon \mathcal{V} \to \mathfrak{U}$, define the induced cochain map $\rho^\#\colon \check{C}^k(\mathfrak{U};\mathcal{F}) \to \check{C}^k(\mathcal{V};\mathcal{F})$ by

$$(\rho^\#\gamma)(V_0,\dots,V_k) = \gamma(\rho V_0,\dots,\rho V_k).$$

   (a) Prove that $\rho^\#$ commutes with δ. Thus we can define an induced cohomology map $\rho^*\colon \check{H}^k(\mathfrak{U};\mathcal{F}) \to \check{H}^k(\mathcal{V};\mathcal{F})$ by $\rho^*[\gamma] = [\rho^\#\gamma]$.

   (b) Complete the proof that ρ^* depends only on the covers $\mathfrak{U}$ and $\mathcal{V}$, and not on the refining map ρ, as follows. Given two refining maps $\rho, \tilde\rho\colon \mathfrak{U} \to \mathcal{V}$, define a map $\theta\colon \check{C}^k(\mathfrak{U};\mathcal{F}) \to \check{C}^{k-1}(\mathcal{V};\mathcal{F})$ by

$$(\theta c)(V_0,\dots,V_{k-1}) = \sum_{i=0}^{k-1}(-1)^i c(\rho V_0,\dots,\rho V_i,\tilde\rho V_i,\dots,\tilde\rho V_{k-1})$$

   and show that $\theta\delta + \delta\theta = \tilde\rho^\# - \rho^\#$.

   (c) For any sheaf homomorphism $\varphi\colon \mathcal{E} \to \mathcal{F}$, show that $\varphi_*\rho^* = \rho^*\varphi_*$.

5. Let M be a smooth manifold, $E \to M$ a vector bundle, and $\mathcal{E}$ the sheaf of germs of C^∞ sections of E. For any open subset $U \subset M$, we have used the notation $\mathcal{E}(U)$ to denote both the space of continuous sections of $\mathcal{E}$ over U and the space of smooth sections of E over U.

   (a) Show that these spaces are isomorphic via the map that sends a smooth section $f : U \to E$ to the section σ_f of $\mathcal{E}$ over U, where
   $$\sigma_f(x) = [f]_x.$$

   (b) Let σ be a smooth section of E over an open set $U \subset M$. Show that the set of points where σ is nonzero as a section of $\mathcal{E}$ is closed in M, while the set of points where σ is nonzero as a section of E is open in M. If M is connected, does this imply σ is constant if it vanishes somewhere? Explain.

6. In some treatments of sheaf theory, the spaces of sections $\mathcal{F}(U)$ are treated as the primary objects instead of the topological space $\mathcal{F}$. This problem outlines that approach.

   Let R be a commutative ring with unit. Define a ***presheaf*** on a manifold M as an assignment of an R-module $\mathcal{F}(U)$ to each open subset $U \subset M$, together with homomorphisms (called "restriction maps") $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ whenever $V \subset U$, satisfying $\rho_{UU} = id_U$ and $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ whenever $W \subset V \subset U$.

   (a) Show that every sheaf gives rise to a presheaf in a canonical way.

   (b) By mimicking the construction of the sheaf of germs of C^∞ functions, show that every presheaf gives rise to a sheaf such that elements of $\mathcal{F}(U)$ can naturally be identified with sections of $\mathcal{F}$ over U. (This operation is sometimes given the unwieldy name of "sheafification functor.")

   (c) Suppose $\mathcal{F}$ is a sheaf. Show that the sheafification of the presheaf associated with $\mathcal{F}$ is canonically isomorphic to $\mathcal{F}$.

7. Show that the converse of Problem 6c above is not true, that is, there are presheaves that are not equal to the spaces of sections of their sheafifications. Formulate necessary and sufficient conditions on a presheaf so that equality will hold. (Presheaves satisfying these conditions are called "complete.")