Math 548 **Geometric Structures** **Winter 1997**

SUPPLEMENTARY ASSIGNMENT

1. Let $E \to M$ be a k-dimensional real vector bundle with a fiber metric, and let $SO(E) \to M$ denote the corresponding principal $SO(k)$ bundle. Let $\mathfrak{U} = \{U_\alpha\}$ be a trivializing cover for $SO(E)$, such that all the sets U_α are convex. On each nonempty intersection $U_\alpha \cap U_\beta$, let $\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to SO(k)$ denote the transition function, and choose a lift $\tilde{\tau}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to Spin(k)$.

   (a) On each nonempty triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, define

   $$\omega_{\alpha\beta\gamma}(x) = \tilde{\tau}_{\alpha\beta}(x)\tilde{\tau}_{\beta\gamma}(x)\tilde{\tau}_{\gamma\alpha}(x).$$

   Show that $\omega \in \check{C}^2(\mathfrak{U}; \mathbf{Z}_2)$ and $\delta\omega = 0$.

   (b) Show that the cohomology class $w_2(E) = [\omega] \in \check{H}^2(M; \mathbf{Z}_2)$ is independent of choices (of cover, local trivializations, lifts, or fiber metric).

   (c) Show that w_2 is natural, and is therefore a characteristic class. It is called the ***second Stiefel-Whitney class*** of E.

   (d) Show that M admits a spin structure if and only if $w_2(TM) = 0$.