Math 548 **Geometric Structures** **Spring 1996**

ASSIGNMENT #2

DUE 6/7/96

1. Let G be a subgroup of $GL(n,\mathbf{R})$. Recall that a ***G-structure*** on an n-manifold M is a reduction of the structure group of TM to G. A G-structure is said to be ***integrable*** if, in a neighborhood of every point, there exist coordinates $\{x^i\}$ such that $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ is a G-frame. Show that the following pairs of structures are equivalent.
   (a) Integrable $O(n)$-structures and flat Riemannian metrics.
   (b) Integrable $GL(k,\mathbf{C})$-structures and complex structures, where $n = 2k$. [By identifying $\mathbf{R}^{2k}$ with $\mathbf{C}^k$ via the correspondence $(x^1, y^1, \dots, x^k, y^k) \leftrightarrow (x^1+iy^1, \dots, x^k+iy^k)$, we can consider $GL(k,\mathbf{C})$ as a subgroup of $GL(2k,\mathbf{R})$. A ***complex structure*** on a $2k$-dimensional manifold M is a maximal atlas of coordinate charts taking their values in $\mathbf{C}^k = \mathbf{R}^{2k}$ which overlap holomorphically.]
   (c) Integrable $Sp(k,\mathbf{R})$-structures and symplectic structures, where $n = 2k$. [Here $Sp(k,\mathbf{R})$ is the group of linear transformations of $\mathbf{R}^{2k}$ preserving the 2-form $\sum dx^j \wedge dy^j$. Hint: for one direction, use the Darboux theorem.]
   (d) Integrable G_m structures and m-dimensional foliations, where G_m is the subgroup of $GL(n,\mathbf{R})$ consisting of matrices that map $\mathbf{R}^m$ to itself.
   (e) Integrable $SL(n,\mathbf{R})$ structures and $SL(n,\mathbf{R})$ structures.
2. (a) Show that any principal $\mathbf{S}^1$ bundle over $\mathbf{S}^1$ is trivial.
   (b) Show that there is a nontrivial $\mathbf{S}^1$ bundle over $\mathbf{S}^1$ with structure group $\mathbf{Z}_2$.
3. By adapting the proof given in class for H^1, prove the DeRham theorem for 2-dimensional cohomology: $H^2_{DR}(M) \cong \check{H}^2(M;\mathbf{R})$.
4. For any complex vector bundle $E \to M$, let $c_1(E) \in \check{H}^2(M;\mathbf{Z})$ denote the integral first Chern class. Prove the addition formula
$$c_1(E \oplus \widetilde{E}) = c_1(E) + c_1(\widetilde{E}).$$
[Work directly from the Čech construction given in class; don't use the axioms for Chern classes.]
5. Let $E \to M$ be a k-dimensional (real or complex) vector bundle. Suppose there exists a global section Ω of $\Lambda^k E$, and give E the corresponding structure group $G = SL(k,\mathbf{R})$ or $SL(k,\mathbf{C})$. Prove that a connection ∇ on E is a G-connection if and only if $\nabla\Omega = 0$.
6. Let $P \to M$ be a principal G-bundle over an n-manifold M, and $H \subset TP$ an n-dimensional plane field that is everywhere transverse to the vertical plane field V. A piecewise smooth curve $\sigma\colon I \to P$ is said to be ***horizontal*** if $\dot\sigma(t) \in H_{\sigma(t)}$ for all $t \in I$. (At points where σ is not smooth, this means that both left-handed and right-handed derivatives are in $H_{\sigma(t)}$.) If $\gamma\colon I \to M$ is a piecewise smooth curve in M, σ is called a ***lift*** of γ if $\pi \circ \sigma = \gamma$.

(a) Given any curve $\gamma\colon I \to M$ and any point $p \in P_{\gamma(0)}$, show γ has a unique horizontal lift σ such that $\sigma(0) = p$.

(b) Now suppose H is the horizontal plane field associated to a connection ω in P, E is a vector bundle associated to P, and ∇ is the G-connection in E determined by ω. Show that, if $\sigma\colon I \to P$ is any lift of $\gamma\colon I \to M$, then the frame $(\sigma_1, \ldots, \sigma_k)$ for E along γ determined by σ is parallel if and only if σ is horizontal.

7. Let M be a connected smooth manifold and $x \in M$. Suppose $E \to M$ is a flat bundle (i.e. a vector bundle with a flat connection ∇). For any piecewise smooth closed curve γ starting and ending at x, parallel translation around γ defines a linear map $P_\gamma\colon E_x \to E_x$.

   (a) Show that P_γ depends only on the homotopy class of γ in $\pi_1(M, x)$.

   (b) Show that the map $P\colon \pi_1(M, x) \to GL(E_x)$ so defined is a representation of $\pi_1(M, x)$, called the ***holonomy representation***.

   (c) We say that two flat bundles (E, ∇) and $(\widetilde{E}, \widetilde{\nabla})$ over M are equivalent if there is a bundle isomorphism $F\colon E \to \widetilde{E}$ such that $F^*\widetilde{\nabla} = \nabla$. For any group Γ, two representations $\rho\colon \Gamma \to GL(V)$, $\tilde{\rho}\colon \Gamma \to GL(\widetilde{V})$ are said to be equivalent if there is an isomorphism $\varphi\colon V \to \widetilde{V}$ such that $\varphi \circ \rho(g) = \tilde{\rho}(g) \circ \varphi$ for all $g \in \Gamma$. Show that the holonomy representation gives a one-to-one correspondence between equivalence classes of finite-dimensional representations of $\pi_1(M, x)$ and equivalence classes of flat bundles over M.

8. (a) Prove that the Maurer-Cartan form ξ on any Lie group satisfies the following identity, known as the ***Maurer-Cartan equation***:

$$d\xi + \xi \wedge \xi = 0.$$

   (b) If $P = M \times G \to M$ is the canonical trivial principal G-bundle over M, show that $\pi_2^*\xi$ is a flat connection on P, where π_2 is projection onto G and ξ is the Maurer-Cartan form of G.