Math 548 **Geometric Structures** **Spring 1996**

ASSIGNMENT #1

DUE 4/19/96

REMINDER: There will be no class Friday, April 4.

1. Suppose $E \to M$ is a vector bundle with a fiber metric and $S \subset E$ is a subbundle.
   (a) Show that $S^\perp$ is a vector subbundle of E.
   (b) Show that $E \cong S \oplus S^\perp$.
2. (a) Suppose $E_1 \to M_1$, $E_2 \to M_2$ are vector bundles, and $\Phi\colon E_1 \to E_2$ is a vector bundle homomorphism with constant rank. Show that $\operatorname{Ker}\Phi \subset E_1$ and $\operatorname{Im}\Phi \subset E_2$ are vector subbundles.
   (b) If $E \to M$ is an m-dimensional vector bundle and $S \subset E$ a k-dimensional vector subbundle, define the ***quotient bundle*** E/S as the set
   $$E/S := \coprod_{p\in M} E_p/S_p.$$
   Show that E/S has a unique structure as an $(m-k)$-vector bundle such that the natural projection $E \to E/S$ is a vector bundle homomorphism with kernel S.
   (c) If E is a vector bundle with fiber metric and $S \subset E$ is a vector subbundle, show that E/S is canonically isomorphic to $S^\perp$.
   (d) Let $\widetilde{M}$ be a Riemannian manifold and $M \subset \widetilde{M}$ an (immersed or embedded) submanifold. Show that the ***algebraic normal bundle*** $NM := (T\widetilde{M}|_M)/TM$ and the ***geometric normal bundle*** $N_gM := TM^\perp \subset T\widetilde{M}|_M$ are canonically isomorphic.
3. A ***line bundle*** is a 1-dimensional vector bundle. For M a smooth manifold, let $V^1_{\mathbf{R}}(M)$ denote the set of isomorphism classes of real line bundles over M, and $V^1_{\mathbf{C}}(M)$ the set of isomorphism classes of complex line bundles.
   (a) Show that $V^1_{\mathbf{R}}(M)$ and $V^1_{\mathbf{C}}(M)$ are abelian groups under tensor product. What is the identity? What is the inverse of a bundle E?
   (b) Show every element of $V^1_{\mathbf{R}}(M)$ has order at most 2.
   (c) Compute $V^1_{\mathbf{R}}(S^1)$.
4. (BUNDLE CONSTRUCTION THEOREM) Let M, F be smooth manifolds, G a Lie group, and $\theta\colon G \times F \to F$ a smooth group action, which we write $\theta(g,f) = g\cdot f$. Suppose we are given a covering of M by open sets $\mathfrak{U}_\alpha$, and for each pair $\mathfrak{U}_\alpha, \mathfrak{U}_\beta$ a smooth map $\tau_{\alpha\beta}\colon \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta \to G$, called a ***transition function***, such that the following ***cocycle condition*** is satisfied:
   $$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \qquad \text{for all } p \in \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta \cap \mathfrak{U}_\gamma.$$

(a) Show that there is a fiber bundle $E \to M$ with fiber F and structure group G, such that each set $\mathfrak{U}_\alpha$ has a local trivialization $\varphi_\alpha \colon \pi^{-1}(\mathfrak{U}_\alpha) \to \mathfrak{U}_\alpha \times F$ whose overlaps are given by

$$\varphi_\alpha \circ \varphi_\beta^{-1}(p, f) = (p, \tau_{\alpha\beta}(p) \cdot f).$$

[Hint: define E as a suitable quotient space of $\coprod_\alpha \mathfrak{U}_\alpha \times F$.]

(b) Suppose we are given two sets of transition functions $\{\tau_{\alpha\beta}\}$, $\{\tilde{\tau}_{\alpha\beta}\}$ defined on the same collection of open sets $\{\mathfrak{U}_\alpha\}$, and satisfying the cocycle condition. Show that they determine isomorphic bundles if and only if there exist smooth maps $\lambda_\alpha \colon \mathfrak{U}_\alpha \to G$ such that

$$\tilde{\tau}_{\alpha\beta}(p) = \lambda_\alpha(p)^{-1}\tau_{\alpha\beta}(p)\lambda_\beta(p), \qquad \text{for all } p \in \mathfrak{U}_\alpha \cap \mathfrak{U}_\beta.$$

5. Cover the sphere $\mathbf{S}^2 = \mathbf{CP}^1$ with two open sets $\mathfrak{U}_j := \{[z^1, z^2] : z^j \neq 0\}$. Define coordinate charts $z \colon \mathfrak{U}_2 \to \mathbf{C} = \mathbf{R}^2$ and $\zeta \colon \mathfrak{U}_1 \to \mathbf{C} = \mathbf{R}^2$ by

$$z[z^1, z^2] = \frac{z^1}{z^2};$$
$$w[z^1, z^2] = \frac{z^2}{z^1}.$$

(a) By means of the identifications $T\mathbf{S}^2|_{\mathfrak{U}_j} \leftrightarrow T\mathfrak{U}_j \leftrightarrow T\mathbf{C} \leftrightarrow \mathbf{C} \times \mathbf{C}$ given by these charts, show that $T\mathbf{S}^2$ can be given the structure of a complex line bundle and compute its transition function τ_{12}. [See Problem 4.]

(b) Compute the transition function $\tilde{\tau}_{12}$ for the universal bundle $\mathbf{U} \to \mathbf{CP}^1$ with respect to the same charts.

(c) Prove that $T\mathbf{S}^2 \cong \mathbf{U}^* \otimes \mathbf{U}^*$.

(d) Prove that $\mathbf{U} \not\cong \mathbf{U}^*$. [Hint: use the fact that $T\mathbf{S}^2$ does not admit a global nonvanishing section.]