Math 548 | **Geometric Structures** | **Winter 2012**

ASSIGNMENT #4
DUE 5:00PM MARCH 12, 2012

1. (a) Let G be a Lie group, and let θ be its Maurer–Cartan form. Prove that θ satisfies the following identity, known as the ***Maurer-Cartan equation***:

$$d\theta + \tfrac{1}{2}[\theta,\theta] = 0.$$

   (b) If $P = M \times G \to M$ is a product principal G-bundle over M, show that $\pi_2^*\theta$ is a flat connection on P, where π_2 is projection onto G and θ is the Maurer–Cartan form of G.

2. Let M be a smooth n-manifold, and let $F(M)$ be the frame bundle of M, considered as a principal $\mathrm{GL}(n,\mathbb{R})$-bundle with projection $\pi\colon F(M) \to M$. A point $f \in F(M)$ is a basis for the tangent space T_xM, where $x = \pi(f) \in M$; such a basis can be identified with a linear isomorphism $f\colon \mathbb{R}^n \to T_xM$. Define an $\mathbb{R}^n$-valued 1-form θ on $F(M)$, called the ***soldering form***, by

$$\theta_f(v) = f^{-1}(d\pi_f(v)), \qquad f \in F(M),\ v \in T_fF(M).$$

   (a) Show that θ is smooth.

   (b) Let ω be a principal connection on $F(M)$, and define a smooth $\mathbb{R}^n$-valued 2-form Θ on $F(M)$ by

$$\Theta = d\theta + \omega \wedge \theta, \qquad \text{i.e.,} \qquad \Theta^i = d\theta^i + \omega^i_j \wedge \theta^j.$$

   Show that

$$d\Theta + \omega \wedge \Theta = \Omega \wedge \theta,$$

   where Ω is the curvature 2-form of ω.

   (c) Let ∇ be the connection on TM corresponding to ω. Show that Θ is identically zero if and only if ∇ is symmetric.

3. Let M be a connected smooth manifold, let $E \to M$ be a smooth rank-k vector bundle, and let ∇ be a connection in E. Choose a basepoint $p \in M$, and for any piecewise smooth loop $\gamma\colon [0,1] \to M$ based at p, let $P(\gamma)\colon E_p \to E_p$ be the linear map defined by parallel transport along γ from $t = 0$ to $t = 1$:

$$P(\gamma)v = \sigma(1),$$

where $\sigma\colon [0,1] \to E$ is the parallel section along γ satisfying $\sigma(0) = v$. Define a subset $H \subseteq \mathrm{GL}(E_p)$ by

$$H = \{P(\gamma) : \gamma \text{ is a piecewise smooth loop based at } p\}.$$

   (a) Show that H is a subgroup of $\mathrm{GL}(E_p)$, called the ***holonomy group of*** ∇ ***at*** p.

   (b) By choosing a basis for E_p, we may identify $\mathrm{GL}(E_p)$ with $\mathrm{GL}(k,\mathbb{R})$. Show that, up to conjugacy, the resulting subgroup $H \subseteq \mathrm{GL}(k,\mathbb{R})$ is independent of choices: If we choose any other point $p' \in M$ and any basis for $E_{p'}$, then the resulting group H' is conjugate in $\mathrm{GL}(k,\mathbb{R})$ to H.

   (c) Show that E admits a reduction to H.

   (d) If E admits a reduction to some subgroup $G \subseteq \mathrm{GL}(k,\mathbb{R})$ and ∇ is a G-connection, show that H is conjugate to a subgroup of G.