**Math 547** **Geometric Structures** **Winter 1996**

ASSIGNMENT #4

DUE 3/15/96

## I. Required problems.

1. [LN] Problem 6.11.
2. [LN] Problem 7.8.
3. Let $M \subset \mathbf{R}^3$ be a compact, embedded 2-manifold with the induced metric.
   (a) Show that $M$ cannot have $K \leq 0$. [Hint: look at a point where the distance from the origin takes a maximum.]
   (b) Show that $M$ cannot have $K \geq 0$ unless $\chi(M) > 0$.
4. Let $M \subset \mathbf{R}^3$ be a surface of revolution as described in Problem 4, Assignment #2.
   (a) Compute the Gaussian curvature of $M$ in terms of $a(t), b(t)$.
   (b) Show that there exists a one-parameter family of mutually non-congruent surfaces of revolution $\{M_t : t \in (-\varepsilon, \varepsilon)\}$ with constant Gaussian curvature $+1$, with $M_0$ equal to (an open subset of) the round unit sphere. (Can you produce them with a hollow rubber ball?)
5. If $(M_1, g_1)$ and $(M_2, g_2)$ are Riemannian manifolds, define the ***product metric*** $g = g_1 \oplus g_2$ on $M_1 \times M_2$ by

$$g(X_1 + Y_1, X_2 + Y_2) = g_1(X_1, Y_1) + g_2(X_2, Y_2),$$

   where $X_i, Y_i \in T_{p_i}M_i$ under the natural identification $T_{(p_1,p_2)}M_1 \times M_2 = T_{p_1}M_1 \oplus T_{p_2}M_2$.
   (a) Show that for each point $p_i \in M_i$, the submanifolds $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally geodesic.
   (b) If $\Pi \subset T(M_1 \times M_2)$ is a 2-plane spanned by $X_1 \in TM_1$ and $X_2 \in TM_2$, show that $K(\Pi) = 0$.
   (c) Show that the product metric on $S^2 \times S^2$ has non-negative sectional curvature.
   (d) Show that there is an embedding of $T^2$ in $S^2 \times S^2$ such that the induced metric is flat.
6. [LN] Problem 8.10.
7. Suppose $(M, g)$ is a Riemannian manifold and $p \in M$. Show that the second-order Taylor series of $g$ in normal coordinates centered at $p$ is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{kl} R_{iklj} x^k x^l + O(|x|^3).$$

   [Hint: Let $\gamma(t) = (tV^1, \ldots, tV^2)$ be a radial geodesic and $J(t) = tW^i \partial_i$ a Jacobi field along $\gamma$, and compute the first four $t$-derivatives of $|J(t)|^2$ at $t = 0$ in two ways.]

8. Suppose $G$ is a Lie group with Lie algebra $\mathfrak{g}$. For any left-invariant vector fields $(X_1, \ldots, X_n)$ that form a basis for $\mathfrak{g}$, define the *structure constants* $c_{ij}^k$ by

$$[X_i, X_j] = \sum_k c_{ij}^k X_k.$$

A metric $g$ on $G$ is said to be ***left-invariant*** if it is preserved by all left multiplications, and ***bi-invariant*** if it is preserved by all left and right multiplications.

(a) Show that a metric $g$ is left-invariant if and only if $g_{ij} = g(X_i, X_j)$ are constants.

(b) For an arbitrary left-invariant metric $g$, compute the Christoffel symbols of the Levi-Civita connection (with respect to the basis $\{X_i\}$) in terms of $c_{ij}^k$ and $g_{ij}$.

(c) If $g$ is bi-invariant, show that

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

(d) Let

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

be a basis for the Lie algebra $SO(3)$. For any positive constant $a$, define a left-invariant metric on $SO(3)$ by letting $X, Y, aZ$ be an orthonormal frame. Compute the sectional curvatures $K(X, Y)$, $K(Y, Z)$, and $K(Z, X)$ of this metric. [The case $a = 1$ is a familiar Riemannian manifold—do you recognize it?]

9. Show that any local diffeomorphism between compact, connected manifolds is a covering map.

## II. Optional problems.

10. A subset $\mathcal{U}$ of a Riemannian manifold $M$ is said to be *convex* if, for each $p, q \in \mathcal{U}$, there is a unique (in $M$) minimizing geodesic from $p$ to $q$ lying entirely in $\mathcal{U}$. Show that every point has a convex neighborhood, as follows.

(a) Let $p \in M$ be fixed, and let $W$ be a totally normal neighborhood of $p$. For $\varepsilon > 0$ small enough that $B_\varepsilon(p) \subset W$, define a subset $\mathcal{W}_\varepsilon \subset TM \times \mathbf{R}$ by

$$\mathcal{W}_\varepsilon = \{(q, V, t) \in TM \times \mathbf{R} : q \in B_\varepsilon(p), V \in T_qM, |V| = 1, |t| < 2\varepsilon\}.$$

Define $f \colon \mathcal{W}_\varepsilon \to \mathbf{R}$ by

$$f(q, V, t) = d(\exp_q(tV), p)^2.$$

Show that $f$ is smooth. [Hint: use normal coordinates centered at $p$.]

(b) Show that, if $\varepsilon$ is chosen small enough, then $\partial^2 f / \partial t^2 > 0$ on $\mathcal{W}_\varepsilon$. [Hint: compute $f(p, V, t)$ explicitly and use continuity.]

(c) If $q_1, q_2 \in B_\varepsilon(p)$ and $\gamma$ is a minimizing geodesic from $q_1$ to $q_2$, then $d(\gamma(t), p)$ attains its maximum at one of the endpoints of $\gamma$.

(d) If $\varepsilon$ is chosen as above, show that $B_\varepsilon(p)$ is convex.

11. Show that every compact smooth 2-manifold has a smooth triangulation. [Hint: using the result of Problem 10, cover $M$ with convex $\varepsilon$-balls, and let the sides of the triangles be minimizing geodesic segments.]

12. Recall that a ***pseudo-Riemannian metric*** on a manifold $M$ is a symmetric covariant 2-tensor field $g$ which is ***nondegenerate*** at each point, which means that $g(X, Y) = 0$ for all $Y$ implies $X = 0$.

(a) Show that the Fundamental Lemma of Riemannian Geometry (Theorem 3.7) holds for pseudo-Riemannian metrics, and define the curvature tensor just as we did for Riemannian metrics.

(b) If $\widetilde{M}$ is a sub-Riemannian manifold, a submanifold $\iota\colon M \subset \widetilde{M}$ is called ***space-like*** if $\iota^* g$ is a Riemannian metric on $M$. Show that the Gauss Formula holds for spacelike submanifolds.

(c) Use this to give another proof that the sectional curvature of $\mathbf{H}^n_R$ is $-1/R^2$.