Math 547 | **Geometric Structures** | **Winter 1996**

ASSIGNMENT #3
DUE 2/23/96

I. Required problems.

1. [LN] Problem 4.5.

2. A curve $\gamma\colon I \to M$ is said to ***approach infinity*** if, for every compact set $K \subset M$, there is a time $T \in I$ such that $\gamma(t) \notin K$ for $t > T$. Prove that a Riemannian manifold is complete if and only if every admissible curve that approaches infinity has infinite length.

3. Show that any homogeneous Riemannian manifold is complete.

4. We now have two kinds of "metrics" on a Riemannian manifold—the Riemannian metric and the distance function. Correspondingly, there are two definitions of "isometry" between Riemannian manifolds—a diffeomorphism that pulls one Riemannian metric back to the other, and a homeomorphism that pulls one distance function back to the other. Prove that these two kinds of isometry are identical. [Hint: for the hard direction, use the exponential map to show the homeomorphism is smooth, and then look at suitable derivatives of the distance function.]

5. Let M be a Riemannian manifold, and $\{x^i\}$ any local coordinates on M.
   (a) Compute the components of the Riemann curvature tensor in terms of the Christoffel symbols in coordinates.
   (b) Now suppose $\{x^i\}$ are normal coordinates centered at $p \in M$. Show that the following holds at p:
$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right).$$

6. If ω is a 1-form, let
$$\omega_{i;jk}dx^i \otimes dx^j \otimes dx^k$$
be the local expression for $\nabla^2\omega$. Prove the ***Ricci identity***
$$\omega_{i;jk} - \omega_{i;kj} = R_{kji}{}^l\omega_l.$$
[Hint: Instead of expanding out the components of $\omega_{i;jk}$ in terms of the Christoffel symbols, try to find an expression for $\nabla^2\omega$ similar to that in Problem 2.14, and use the definition of the curvature operator.]

II. Optional problems.

7. Let ∇ be any linear connection on a manifold M. We can define the ***curvature operator*** of ∇ by the same formula as in the Riemannian case; ∇ is said to be ***flat*** if $R(X,Y)Z \equiv 0$. Prove that the following are equivalent.

(a) ∇ is flat.

(b) Near every point $p \in M$, there exists a local frame which is parallel along every curve.

(c) For all $p, q \in M$, parallel translation along a curve segment γ from p to q depends only on the homotopy class of γ.

(d) For all sufficiently nearby points $p, q \in M$, parallel translation from p to q depends only on p and q, not on the curve segment joining them.

(e) Parallel translation around any sufficiently small closed curve is the identity.

8. Let (M, g) be a Riemannian manifold. I claimed in class that, among all linear connections on M that are compatible with g, the Riemannian connection is the only one whose geodesics are locally minimizing. It turns out this is false, as the following example shows. Define a connection on $\mathbf{R}^3$ by setting

$$\begin{aligned} \Gamma_{12}^3 = \Gamma_{23}^1 = \Gamma_{31}^2 &= 1, \\ \Gamma_{21}^3 = \Gamma_{32}^1 = \Gamma_{13}^2 &= -1, \end{aligned}$$

and all other Christoffel symbols to zero. Show that this connection is compatible with the Euclidean metric and has minimizing geodesics.