Math 547 **Geometric Structures** **Winter 1996**

Assignment #2

Due 2/9/96

I. Required problems.

1. [LN], Problem 3.3.
2. If (M, g) and $(\widetilde{M}, \tilde{g})$ are Riemannian manifolds, a map $\varphi\colon M \to \widetilde{M}$ is a ***local isometry*** if each point $p \in M$ has a neighborhood $\mathcal{U}$ such that $\varphi|_{\mathcal{U}}$ is an isometry onto an open subset of $\widetilde{M}$. Suppose M is connected, and suppose $\varphi, \psi\colon M \to \widetilde{M}$ are local isometries such that $\varphi(p) = \psi(p)$ and $\varphi_* = \psi_*$ at p. Show that $\varphi \equiv \psi$.
3. Consider $\mathbf{R}^n$ as a Riemannian manifold with the Euclidean metric.
   (a) Let $E(n)$ be the set of $(n+1) \times (n+1)$ real matrices of the form
   $$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix},$$
   where $A \in O(n)$ and $b \in \mathbf{R}^n$. Show that $E(n)$ is a closed Lie subgroup of $GL(n+1, \mathbf{R})$.
   (b) Define a map $E(n) \times \mathbf{R}^n \to \mathbf{R}^n$ by identifying $\mathbf{R}^n$ with the subset
   $$S = \{(x, 1) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n\}$$
   of $\mathbf{R}^{n+1}$ and restricting the linear action of $E(n)$ on $\mathbf{R}^{n+1}$ to S. Show that this is a smooth action of $E(n)$ on $\mathbf{R}^n$ by isometries of the Euclidean metric.
   (c) Show that $E(n)$ acts transitively on orthonormal frames for $\mathbf{R}^n$.
   (d) Show that $\mathfrak{I}(\mathbf{R}^n) = E(n)$, $\mathfrak{I}(\mathbf{H}^n_R) = O_+(n, 1)$, and $\mathfrak{I}(\mathbf{S}^n_R) = O(n+1)$.
   (e) Strengthen the result above by showing that if M is one of our model Riemannian manifolds ($M = \mathbf{R}^n$, $\mathbf{H}^n_R$, or $\mathbf{S}^n_R$), U, V are connected open subsets of M, and $\varphi\colon U \to V$ is an isometry, then φ is the restriction to U of an element of $\mathfrak{I}(M)$.
4. Let $\gamma(t) = (a(t), b(t))$, $t \in I$, be a unit-speed curve in the xz-plane, and suppose that $a(t) > 0$ and $b'(t) > 0$ for all $t \in I$. Let $M \subset \mathbf{R}^3$ be the surface of revolution obtained by revolving the image of γ about the z-axis.
   (a) Show that M is an embedded submanifold, and that the map $\varphi(\theta, t) = (a(t)\cos\theta, a(t)\sin\theta, b(t))$ from $\mathbf{R} \times I$ to $\mathbf{R}^3$ determines coordinates (θ, t) for M in a neighborhood of any point.
   (b) Compute the pullback of the metric induced on M by the Euclidean metric in (θ, t) coordinates, and compute its Christoffel symbols.
   (c) Show that each "meridian" $\{\theta = \theta_0\}$ is a geodesic on M.
   (d) Determine necessary and sufficient conditions for a "latitude circle" $\{t = t_0\}$ to be a geodesic.

5. Let $\mathbf{H}^n_R$ denote the n-dimensional hyperbolic space of radius R.
   (a) Determine the unit-speed parametrization of the geodesic in the hyperboloid model starting at $N = (0, \ldots, R)$ with initial tangent vector $\partial/\partial\xi^1$.
   (b) Prove that each geodesic on $\mathbf{H}^n_R$ is defined for all $t \in \mathbf{R}$, and that the image of each geodesic is an entire branch of a great hyperbola.
6. A vector (or tensor) field V is said to be ***parallel*** if $\nabla V \equiv 0$, or equivalently if V is parallel along every curve.
   (a) Let $p \in \mathbf{R}^n$ and $V_p \in T_pM$. Show that V_p has a unique extension to a parallel vector field V on $\mathbf{R}^n$.
   (b) Let $\mathcal{U}$ be the open subset of S^2 on which spherical coordinates (θ, φ) are defined (see Problem 3.11), and let $V = \partial/\partial\varphi$ in these coordinates. Compute $\nabla_{\frac{\partial}{\partial\theta}} V$ and $\nabla_{\frac{\partial}{\partial\varphi}} V$, and conclude that V is parallel along the equator and along each meridian $\theta = \theta_0$.
   (c) Let $p = (0, 0)$ in spherical coordinates. Show that V_p has no parallel extension to any neighborhood of p.
   (d) Use (6a) and (6c) to show that $\mathbf{S}^2$ is not locally isometric to $\mathbf{R}^2$.

II. Optional problems.

7. Let $\mathbf{H}^2$ denote the hyperbolic plane, i.e. hyperbolic space of dimension 2. For the purposes of this problem, we consider $\mathbf{H}^2$ as the upper half-plane in $\mathbf{R}^2$ with the metric $h = (dx^2 + dy^2)/y^2$. Let $SL(2, \mathbf{R})$ denote the group of 2×2 real matrices of determinant 1.
   (a) Considering $\mathbf{H}^2$ as a subset of the complex plane with coordinate $z = x + iy$, let
   $$A \cdot z = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}).$$
   Show that this defines a smooth action of $SL(2, \mathbf{R})$ on $\mathbf{H}^2$ by isometries of the hyperbolic metric.
   (b) You showed in Problem 3 that the isometry group of $\mathbf{H}^2$ is $O_+(2, 1)$. Show that $O_+(2, 1) \cong SL(2, \mathbf{R})/ \pm I$.