Math 547 **Geometric Structures** **Winter 1996**

ASSIGNMENT #1

DUE 1/26/96

The notation **[LN]** refers to the lecture notes. You should make sure you know how to do all the problems in **[LN]** ; the results of those problems may be used freely in doing these problems.

I. Required problems.

1. **[LN]** Problem 2.11.
2. **[LN]** Problem 2.15.
3. **[LN]** Problem 2.27.
4. **[LN]** Problem 2.28.
5. Suppose M is a manifold with a linear connection, $p \in M$, and $X, Y \in \mathcal{T}(M)$. Show that $\nabla_{X_p} Y$ depends only on the values of Y along any curve tangent to X_p. More precisely, suppose that $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is a curve with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$, and suppose Y and $\widetilde{Y}$ are vector fields that agree along γ. Show that $\nabla_{X_p} Y = \nabla_{X_p} \widetilde{Y}$.
6. Let ∇ be a linear connection on M, and define a map $\tau\colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

   (a) Show that τ is a $\binom{2}{1}$-tensor field, called the ***torsion tensor*** of ∇.

   (b) We say ∇ is ***symmetric*** if its torsion vanishes identically. Show that ∇ is symmetric if and only if its Christoffel symbols with respect to any coordinate frame are symmetric: $\Gamma^k_{ij} = \Gamma^k_{ji}$.

   (c) Show that ∇ is symmetric if and only if, for any smooth function $u \in C^\infty(M)$, its covariant Hessian $\nabla^2 u$ is a symmetric 2-tensor.

II. Optional problems.

7. You have already seen another way of taking "directional derivatives of vector fields," the Lie derivative $\mathcal{L}_X Y$.

   (a) Show that the map $\mathcal{L}\colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ is not a connection.

   (b) Show that Lie differentiation does not give a well-defined way to take directional derivatives of vector fields along curves; more precisely, show that Lemma 2.18 is false if ∇ is replaced by the Lie differentiation operator $\mathcal{L}$. (You have to interpret condition (c) by extending $\dot\gamma(t)$ to some neighborhood of $\gamma(t)$.) [Hint: find a vector field that vanishes along a curve but whose Lie derivative along the curve does not vanish.]

8. (a) If ∇^0 and ∇^1 are any two linear connections on M, show that the difference between them defines a $\binom{2}{1}$-tensor field D by

$$D(X,Y) = \nabla^1_X Y - \nabla^0_X Y,$$

called the ***difference tensor***. Thus, if ∇^0 is any linear connection on M, the set of all linear connections is precisely $\{\nabla^0 + D : D \in \mathcal{T}^2_1(M)\}$.

(b) Show that ∇^0 and ∇^1 determine the same geodesics if and only if their difference tensor is antisymmetric, i.e. $D(X,Y) = -D(Y,X)$. [Hint: the geodesic equation involves only expressions of the form $\nabla_X X$.]