Math 547/548/549 **Geometric Structures** **1994–1995**

SYLLABUS

Lectures: MWF 1:30–2:20
Padelford C-401

Instructor: Jack Lee
Padelford C-546
543-1735
E-mail: `lee@math.washington.edu`
Office Hours: Mon. & Wed. 11:30–12:30 or by appointment

Required text (Autumn and Winter):

- M. DoCarmo, *Riemannian Geometry*, Birkhäuser, Boston, 1992.

Other references (on reserve):

- W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd ed.*, Academic Press, Orlando, 1986.
- F. Morgan, *Riemannian Geometry: A Beginner's Guide*, Jones and Bartlett, Boston, 1993.
- M. Spivak, *A Comprehensive Introduction to Differential Geometry, Vols. I–V*, Publish or Perish, Berkeley, 1979.
- J. Cheeger and D. G. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.
- S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vols. I–II*, Interscience, New York, 1963.

General description:

This course introduces the most important concepts and tools of Riemannian geometry. Central to the subject is the concept of *curvature*, which we will study from a variety of points of view. After introducing the definitions and studying local properties of curvature, we will proceed to some of the "big theorems" relating curvature to global properties such as compactness, fundamental groups, and homology groups.

Prerequisites:

Topology and Geometry of Manifolds (three quarters); specifically, the material in Chapters 1–6 of the Boothby text listed above. For the third quarter, *Linear Analysis* is strongly recommended (may be taken concurrently), and a quarter of *Algebraic Topology* would be helpful.

Homework:

A problem set will be assigned approximately every two weeks, due two weeks later. The homework problems on each assignment will be divided into three groups:

1. *Practice problems:* These may include any of the following types of problems: straightforward generalizations of things we've done in class; elementary proofs that "fill in the gaps" from lectures; computations of simple but illuminating examples; intermediate results that will be used in class or in other problems. You are expected to do *all* of the problems in this category. I recommend that you write up the solutions for your notes; but at the very least, you should read each problem carefully and make sure you know exactly how to do it.

2. *Required problems:* These problems are the heart of the course. There will usually be about five problems in this category on each assignment, to be written up and handed in. They will typically include some proofs and some computations of significant examples.

3. *Optional problems:* Some of these problems carry the ideas from the course farther, or lead you through proofs that we don't have time to cover. Some of them are just particularly difficult examples. If you have time, and if you really want to understand the subject, I encourage you to try as many of these as you can, and turn in those you complete successfully. (See also *Grading*, below.) You may turn in optional problems after an assignment's due date, but only if you get permission from me on or before the due date.

The practice and required homework problems represent the minimum work I think you need to do to master this material sufficiently well to pursue a Ph.D. in any field of pure mathematics that requires a working knowledge of Riemannian geometry. If you plan to do research in differential geometry, you should do all the optional problems as well.

You are invited to work together in study groups on the homework problems (it's usually the best and fastest way to learn). However, when you write up your solutions to hand in, *you must write your own solutions in your own words.* This means, in particular, that you may not appoint one person from your study group to write up the solutions, and then have everyone else copy them down.

Grading:

Your grade will be based on the required and optional homework problems. Roughly speaking, getting 90% or more of the required homework problems correct will get you a 3.8, 70% will get you a 3.0, and scores in between will be linearly interpolated. Optional problems handed in will be counted if they are mostly correct, and may increase your grade by up to 0.4, to a maximum of 4.0. A grade lower than 3.0 means you are not performing at the level expected of a graduate student.

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COURSE OUTLINE

AUTUMN: Riemannian Manifolds, Connections, Curvature, and Geodesics

- Review of Riemannian metrics and their basic properties.
- Affine connections; the Levi-Civita connection on a Riemannian manifold.
- Definition of geodesics; local minimizing properties of geodesics; normal coordinates.
- Geometry of Riemannian submanifolds; the second fundamental form.
- Various notions of curvature and their geometric interpretations.
- Jacobi fields and conjugate points.
- Riemannian manifolds as metric spaces; geodesic and metric completeness.
- Manifolds of constant curvature.

WINTER: Curvature and Topology

- The Gauss-Bonnet theorem: *If M is a compact oriented Riemannian 2-manifold with Gaussian curvature K, then $\int_M K\, dA = 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M, a topological invariant.*
- The Bonnet-Meyers Theorem: *If M is a complete Riemannian manifold with Ricci curvature bounded below by $\kappa > 0$, then M is compact, with finite fundamental group and diameter at most equal to that of the sphere with constant Ricci curvature κ.*
- The Cartan-Hadamard Theorem: *If M is a complete Riemannian manifold with nonpositive sectional curvature, the universal cover of M is diffeomorphic to $\mathbf{R^n}$.*
- The Sphere Theorem: *If M is a compact, simply connected Riemannian manifold whose sectional curvature K satisfies $\frac{1}{4}C < K \le C$, then M is homeomorphic to a sphere.*

SPRING: Hodge/deRham Theory

- The deRham Theorem: *The deRham cohomology spaces of closed differential forms modulo exact forms on a smooth manifold are naturally isomorphic to its real singular homology spaces.*
- Introduction to partial differential operators on manifolds: ellipticity, elliptic regularity, Fredholm theory.
- The Hodge Theorem: *If M is a compact Riemannian manifold, then the k-th deRham cohomology space of M is naturally isomorphic to the space of harmonic k-forms.*
- The Bochner Vanishing Theorem: *If M is a compact Riemannian manifold with strictly positive Ricci curvature, then its first real cohomology vanishes.*