Math 549 **Geometric Structures** **Spring 1995**

ASSIGNMENT #2

DUE WEDNESDAY, 6/7/95

I. Practice problems.

1. Prove the following Ricci identity for covariant 2-tensor fields on a Riemannian manifold:

$$T_{ij,kl} - T_{ij,lk} = \sum_p R^p{}_{ikl} T_{pj} + \sum_p R^p{}_{jkl} T_{ip}.$$

Generalize to covariant tensor fields of any rank.

2. *This problem introduces the theory of integration on non-orientable manifolds.* Let V be an n-dimensional vector space, and let $\Lambda^k V$ denote the space of alternating k-vectors, i.e. k-multilinear maps $V^* \times \cdots \times V^* \to \mathbf{R}$. A *density* on V is a map $\mu \colon \Lambda^n V \to \mathbf{R}$ which is absolutely homogeneous, in the sense that

$$\mu(\lambda X_1 \wedge \cdots \wedge X_n) = |\lambda| \mu(X_1 \wedge \cdots \wedge X_n)$$

for any $X_i \in V$, $\lambda \in \mathbf{R}$. Let ΩV denote the set of densities on V.

   (a) If $\omega \in \Lambda^n V^*$ is an n-covector on V, define a map $|\omega| \colon \Lambda^n V \to \mathbf{R}$ by

$$|\omega|(X_1 \wedge \cdots \wedge X_n) = |\omega(X_1, \ldots, X_n)|.$$

   Show that $|\omega|$ is a density.

   (b) Show that ΩV is a one-dimensional vector space, spanned by $|\omega|$ for any non-zero element $\omega \in \Lambda^n V^*$.

   (c) Now let M be any manifold, not necessarily orientable, and let ΩM denote the bundle of densities on M, i.e. the real line bundle whose fiber at x is $\Omega T_x M$. A *density field* on M is a smooth section of ΩM. If μ is any density field on M, and f is a compactly supported, locally integrable function on M, define $\int_M f\mu$ as follows. Choose a covering of M by coordinate charts $\{\mathcal{U}_i\}$, and a subordinate partition of unity $\{\varphi_i\}$. Define

$$\int_M f\mu = \sum_i \int_{\mathcal{U}_i} \varphi_i(x) f(x) m(x) dx^1 \ldots dx^n,$$

   where $m(x)|dx^1 \wedge \cdots \wedge dx^n|$ is the local coordinate expression for μ in $\mathcal{U}_i$. Show that this definition is independent of choices.

   (d) If M is a Riemannian manifold, orientable or not, show that M has a unique density field dV_g, called the *Riemannian density*, such that $dV_g(e_1 \wedge \cdots \wedge e_n) = 1$ for any local orthonormal frame $(e_1, \ldots, e_n)$.

3. Let Δ denote the Laplace-Beltrami operator on a compact manifold M.

   (a) If $f \in C^\infty(M)$, show that $\Delta u = f$ has a solution $u \in C^\infty(M)$ if and only if $\int_M f \, dV_g = 0$.

(b) Discuss the solvability of $\Delta^2\omega = \eta$ on k-forms.

II. Required problems.

4. Let $P\colon \Gamma(E) \to \Gamma(F)$ be an elliptic partial differential operator of order m on a compact manifold M. Define a linear operator $N\colon L^2(F) \to L^2(E)$ by recalling that $L^2(F) = \operatorname{Im} P \oplus \operatorname{Ker} P^*$, and letting Nf be the unique solution $u \in D_P \cap (\operatorname{Ker} P)^\perp$ to $Pu = f$ when $f \in \operatorname{Im} P$, and $Nf = 0$ when $f \in \operatorname{Ker} P^*$. N is called the *Neumann operator* (or sometimes the *Green's operator*) for P.
   (a) Show that $NPv + Hv = v$ for all $v \in D_P$, and $PNv + Kv = v$ for all $v \in L^2(F)$, where H, K are the orthogonal projections onto $\operatorname{Ker} P$ and $\operatorname{Ker} P^*$, respectively.
   (b) Show that, for each integer $k \geq 0$, N restricts to a bounded operator from $H^k(F)$ to $H^{k+m}(E)$.
   (c) Show that $N\colon L^2(F) \to L^2(E)$ is a compact operator.
   (d) Show that N commutes with both H and P (on appropriate domains).
   (e) In the special case $P = \Delta$ on $\Gamma(\Lambda^k M)$, show that N also commutes with both d and d'.
   (f) If ω is a closed k-form on M, derive a formula for its harmonic representative in terms of N, d, and d', and prove that it works.

5. Let M be a non-orientable, smooth, compact Riemannian manifold (possibly with boundary), and dV_g its Riemannian density (see Problem 2). Prove that the divergence theorem holds on M: $\int_M \delta X \, dV_g = \int_{\partial M} \langle X, N\rangle \, dV_{\tilde g}$, where $\tilde g$ is the induced metric on ∂M and N is the outward unit normal. [Hint: consider the orientable double cover.]

6. *This problem outlines an alternate approach to computing the symbol of the Laplace-Beltrami operator on an oriented manifold.* Suppose V is an n-dimensional vector space endowed with an inner product and an orientation. Let $\{e_i\}$ be any oriented orthonormal basis for V and $\{e^i\}$ the dual basis. Let μ denote the volume element determined by the inner product and the orientation (thus $\mu = e^1 \wedge \cdots \wedge e^n$). In terms of this basis, define a linear operator $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$, the *Hodge star operator*, by setting

   $$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \operatorname{sgn}(\sigma) e^{j_1} \wedge \cdots \wedge e^{j_{n-k}},$$

   where $\sigma \in S_n$ is a permutation and $(j_1, \ldots, j_{n-k})$ is the unique increasing multi-index of length $n-k$ such that

   $$\sigma(i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) = (1, \ldots, n);$$

   and then extend $*$ to all of $\Lambda^k V^*$ by linearity.
   (a) Show that, for any k-form ω, $*\omega$ is the unique $(n-k)$-form such that $\eta \wedge *\omega = \langle \eta, \omega\rangle \, \mu$ for all k-forms η. This proves that $*$ is well-defined independently of choice of basis.

(b) Prove that $**\omega = (-1)^{k(n-k)}\omega$ for all k-forms ω.

(c) Let M be an oriented compact Riemannian manifold without boundary. Show that the (Hodge) L^2 inner product on $\Gamma(\Lambda^k)$ can be written

$$(\omega, \eta) = \int_M \omega \wedge *\eta.$$

(d) Using Stokes's theorem, show that the formal adjoint of d is given by $d'\eta = (-1)^{n(k+1)+1} * d * \eta$ when η is a k-form.

(e) If ξ is a 1-form and ω is a k-form, prove that

$$*(\xi \wedge \omega) = (-1)^k \xi \vee *\omega.$$

[Hint: it might help to prove the relation

$$\xi \vee (\omega \wedge \eta) = (\xi \vee \omega) \wedge \eta + (-1)^k \omega \wedge (\xi \vee \eta),$$

and use part (a).]

(f) Prove that the principal symbol of the Laplace-Beltrami operator is $\delta(x,\xi)\omega = -|\xi|^2\omega$.

7. (a) Show that the Laplace-Beltrami operator commutes with the Hodge star operator (see Problem 6) on k-forms.

(b) Using the Hodge star operator and the Hodge theorem, prove the *Poincaré duality theorem* for smooth manifolds: if M is a smooth, compact, oriented n-manifold, then for any $0 \le k \le n$, the map $P\colon H^k_{dR}(M) \to (H^{n-k}_{dR}(M))^*$ given by

$$P[\omega][\eta] = \int_M \omega \wedge \eta$$

is an isomorphism. Thus $\dim H^k_{dR}(M) = \dim H^{n-k}_{dR}(M)$.

8. Using the Bochner vanishing theorem, show that any Riemannian metric with non-negative Ricci curvature on the n-torus $T^n = S^1 \times \cdots \times S^1$ is flat.

9. Let M be a compact Riemannian manifold, and let λ_1 denote the lowest non-zero eigenvalue of the Laplace-Beltrami operator on functions. By applying Bochner's formula to the 1-form df for an eigenfunction f, prove the following theorem of A. Lichnerowics: *If M is a compact Riemannian n-manifold with Ricci curvature satisfying $Rc(X,X) \ge (n-1)k|X|^2$, then $\lambda_1 \ge nk$.*

10. Work out Bochner's formula for 2-forms and prove it. [If you have some experience with Mathematica, you might try using my `Ricci` package for doing the index computations. This is a Mathematica package for doing arbitrary computations with indexed and non-indexed tensor fields in differential geometry. See me for information if you're interested.]

11. Let M be a compact Riemannian manifold, E a vector bundle over M with a fiber metric and compatible connection, and $P\colon \Gamma(E) \to \Gamma(E)$ a second-order, strongly elliptic, self-adjoint linear partial differential operator. Let $\{\sigma_i\}$ be an orthonormal basis for $L^2(E)$ consisting of P-eigensections, with corresponding eigenvalues $\lambda_i \nearrow +\infty$.

(a) Show that a section $u \in L^2(E)$ is in $H^k(E)$ if and only if

$$\sum_{i=1}^{\infty} \lambda_i^k (u, \sigma_i)^2 < \infty.$$

[Hint: The case $k = 2$ is easy. For $k = 1$, use Gårding's inequality. The general case follows from these two.]

(b) Now specialize to the case in which $M = S^1$ is the unit circle and P is the Laplacian acting on functions, and interpret (11a) in terms of Fourier series.

III. Optional problems.

12. If M is any smooth compact n-manifold (without a given Riemannian metric), and $\alpha \in \mathbf{R}$, define the bundle of α-densities $\Omega^\alpha M$ to be the bundle whose fiber at a point $x \in M$ is the vector space

$$\begin{aligned}\Omega^\alpha T_x M := \{&\text{maps } \mu\colon \Lambda^n T_x M \to \mathbf{R} \text{ such that} \\ &\mu(\lambda X_1 \wedge \cdots \wedge X_n) = |\lambda|^\alpha \mu(X_1 \wedge \cdots \wedge X_n) \text{ for all } \lambda \in \mathbf{R}\}.\end{aligned}$$

(a) If μ is an α-density and ν a β-density, show that $\mu \cdot \nu$, defined by

$$\mu \cdot \nu(X_1 \wedge \cdots \wedge X_n) = \mu(X_1 \wedge \cdots \wedge X_n)\nu(X_1 \wedge \cdots \wedge X_n),$$

is an $\alpha + \beta$-density.

(b) Show that the formula

$$(\mu, \nu) = \int_M \mu \cdot \nu$$

defines a natural inner product on $\Gamma(\Omega^{1/2} M)$. Thus the space of L^2 $\frac{1}{2}$-densities is a "natural" L^2 space on a manifold, which can be defined without reference to a Riemannian metric.

13. Derive the general form of Bochner's formula on k-forms. [Hint: The following notation will probably come in handy. For any multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$, let $\varepsilon^I_J = \operatorname{sgn}(\sigma)$ if $I = \sigma J$ and 0 otherwise. Writing a k-form ω as

$$\omega = {\sum_J}' \omega_J dx^J$$

where $dx^J = dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ and the prime means summation over increasing multi-indices only, you can verify that

$$\begin{aligned} d\omega &= (-1)^k {\sum_{IJ}}' \sum_k \varepsilon^{Jk}_I \omega_{J,k} dx^I, \\ d'\omega &= (-1)^k {\sum_{KL}}' \sum_j \varepsilon^K_{Lj} \omega_{K,}{}^j dx^L \end{aligned}$$

From here on, it's just a messy computation.]