Math 549 **Geometric Structures** **Spring 1995**

ASSIGNMENT #1

DUE WEDNESDAY, 5/3/95

I. Practice problems.

1. Let H be a (real or complex) Hilbert space. Prove that every continuous linear functional on H is of the form $f(x) = (x, y)$ for a unique element $y \in H$.
2. For any open set $\Omega \subset \mathbf{R}^n$, show that compactly supported elements of $H^k(\Omega)$ are in $\mathring{H}^k(\Omega)$.

II. Required problems.

3. Compute the formal adjoint of each of the following differential operators. (In each case, the metric involved is the standard Euclidean one.)
   (a) $d\colon \Gamma(\Lambda^0\mathbf{R}^n) \to \Gamma(\Lambda^1\mathbf{R}^n)$.
   (b) $\nabla\colon \Gamma(\mathcal{T}^1\mathbf{R}^n) \to \Gamma(\mathcal{T}^2\mathbf{R}^n)$.
   (c) $P\colon C^\infty(S^1) \to C^\infty(S^1)$, given by $Pf(\theta) = f'(\theta)\cos\theta$.
4. Let M be a manifold, E, F vector bundles over M, and $P\colon \Gamma(E) \to \Gamma(F)$ a partial differential operator. Considering P as an unbounded operator from $L^2(E)$ to $L^2(F)$, part of showing that P is Fredholm is showing that it has closed image. This problem outlines a proof that $\operatorname{Im} P$ is closed in $L^2(F)$ iff P satisfies the following *a priori* estimate: *there is a constant C such that*
$$\|u\|_{L^2} \le C\|Pu\|_{L^2}$$
for every $u \in L^2(E)$ such that $Pu \in L^2(F)$ and u is orthogonal to $\operatorname{Ker} P$.
   (a) Prove the 'if' part.
   (b) Show that the graph of P is a closed subset of $L^2(E) \times L^2(F)$.
   (c) Let $D_P \subset L^2(E)$ denote the domain of P. The *graph norm* on D_P is
$$\|u\|_P^2 = \|u\|_{L^2}^2 + \|Pu\|_{L^2}^2.$$
   Show that D_P is a Hilbert space in the graph norm.
   (d) Let $Z \subset L^2(E)$ denote the L^2-orthogonal complement of $\operatorname{Ker} P$. Show that $Z \cap D_P$ is the orthogonal complement of $\operatorname{Ker} P$ in D_P (with respect to the graph inner product).
   (e) Prove the 'only if' part. [Hint: consider $P\colon Z \cap D_P \to \operatorname{Im} P$ and look up the open mapping theorem.]
5. Let $P\colon C^\infty(0,1) \to C^\infty(0,1)$ be the differential operator $Pu(x) = u(x) + 2xu'(x)$. Show that P does not have closed image in $L^2(0,1)$. [Hint: observe that P has no L^2 kernel, but the function $u(x) = x^{-1/2}$ is in $\operatorname{Ker} P$ and 'almost' in L^2. Replacing $x^{-1/2}$ by a constant near 0, obtain a sequence of continuous functions u_ε that contradict the estimate of Problem 4.]

6. Show that $C_c^\infty(\mathbf{R}^n)$ is dense in the Sobolev space $H^k(\mathbf{R}^n)$, as follows.
   (a) Using cutoffs, show that compactly supported elements of $H^k(\mathbf{R}^n)$ are dense.
   (b) Choose $\rho \in C_c^\infty(\mathbf{R}^n)$ such that $\int_{\mathbf{R}^n} \rho(x)\,dx = 1$, and set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$. (See Problem 9.) If $u \in H^k(\mathbf{R}^n)$ is compactly supported, show that $u_\varepsilon := u * \rho_\varepsilon \in C_c^\infty(\mathbf{R}^n)$.
   (c) If $u \in C_c^0(\mathbf{R}^n)$, show that $u_\varepsilon \to u$ uniformly.
   (d) Show that the family of operators T_ε defined by $T_\varepsilon u = u * \rho_\varepsilon$ is uniformly bounded in L^2 norm as $\varepsilon \to 0$, i.e. for some constant independent of ε and u,
   $$\|T_\varepsilon u\|_{L^2} \leq C\|u\|_{L^2}.$$
   (e) Using (6c) and (6d) above, together with the fact that $C_c^0(\mathbf{R}^n)$ is dense in $L^2(\mathbf{R}^n)$ (by standard measure theory), and interchanging limits appropriately, show that $u_\varepsilon \to u$ in L^2 norm.
   (f) Using Problem 9, show that, if $u \in H^k(\mathbf{R}^n)$ and $|I| \leq k$, $\partial_I u_\varepsilon \to \partial_I u$ in L^2.
   (g) Prove the result.
7. Give a counterexample to the result of Problem 6 when $\mathbf{R}^n$ is replaced by the unit ball in $\mathbf{R}^n$.

III. Optional problems.

8. Let $P\colon H_1 \to H_2$ be a (possibly unbounded) linear operator between Hilbert spaces. Recall that P is *Fredholm* if it has finite-dimensional kernel and cokernel and closed image. If P is a *closed operator* (i.e. graph(P) is closed in $H_1 \times H_2$), show that requiring closed image is redundant. [Hint: first assume P is bounded; for the general case, consider D_P with the graph norm as in Problem 4.]
9. If u and v are locally integrable functions on $\mathbf{R}^n$, their *convolution product* is
   $$u * v(x) = \int_{\mathbf{R}^n} u(x-y)v(y)dy,$$
   whenever it converges.
   (a) Suppose that $u \in L^p(\mathbf{R}^n)$ and $v \in L^q(\mathbf{R}^n)$, with $1 \leq p, q \leq \infty$, $1/p + 1/q \geq 1$. Show that $u * v$ is defined almost everywhere and is in $L^r(\mathbf{R}^n)$, where $1/r = 1/p + 1/q - 1$, and prove *Young's inequality*:
   $$\|u * v\|_{L^r} \leq \|u\|_{L^p}\|v\|_{L^q}.$$
   [Hint: Since $C_c^0(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$, it suffices to prove Young's inequality for continuous, compactly supported functions. Write
   $$|u(x-y)v(y)| = (|u(x-y)|^p|v(y)|^q)^{1/r}(|u(x-y)|^p)^{1/p-1/r}(|v(y)|^q)^{1/q-1/r};$$
   note that $1/r + (1/p - 1/r) + (1/q - 1/r) = 1$ and use Hölder's inequality.]
   (b) If u and v satisfy the hypotheses of (9a), show that $u * v = v * u$.
   (c) If $u \in C_c^\infty(\mathbf{R}^n)$ and $v \in L^p(\mathbf{R}^n)$ for some $1 \leq p \leq \infty$, use the Dominated Convergence Theorem to show that $\partial(u * v)/\partial x^j = (\partial u/\partial x^j) * v$.
   (d) Show that the conclusion of (9c) continues to hold if $u \in H^1(\mathbf{R}^n)$ and $v \in C_c^\infty(\mathbf{R}^n)$.