Math 548 **Geometric Structures** **Winter 1995**

Assignment #4

Due Wednesday, 3/15/95

I. Practice problems.

1. Let $\Delta \subset \mathbf{R}^N$ be a k-simplex. The notation $\operatorname{Int}\Delta$ means the open simplex with the same vertices as Δ.
   (a) Show that specifying an orientation for Δ (i.e. an equivalence class of orderings of its vertices) is equivalent to specifying an orientation for $\operatorname{Int}\Delta$ (as a smooth k-manifold).
   (b) Observe that the union of $\operatorname{Int}\Delta$ with the interiors of its $(k-1)$-dimensional faces is a manifold-with-boundary. Prove that the $(k-1)$-chain $\partial\Delta$ consists of the sum of the $(k-1)$-faces with their Stokes orientations.

II. Required problems.

2. Let K be any simplicial complex.
   (a) Show that $C^k_\Delta(K;\mathbf{R})$ is canonically isomorphic to $(C_{k,\Delta}(K;\mathbf{R}))^*$.
   (b) Show that $H^k_\Delta(K;\mathbf{R})$ is canonically isomorphic to $(H_{k,\Delta}(K;\mathbf{R}))^*$.

   (In this context, *canonical* just means independent of choices.)
3. Let M be a compact, oriented, smooth n-manifold, $K \subset \mathbf{R}^N$ a simplicial complex, and $\tau\colon |K| \to M$ a smooth triangulation. For any real k-chain $c = \sum a_i\Delta_i \in C_{k,\Delta}(K;\mathbf{R})$, and any smooth k-form $\omega \in \Gamma(\Lambda^k M)$, define

$$\int_c \omega = \sum a_i \int_{\Delta_i} \tau^*\omega.$$

   Define a map $\mathcal{I}\colon \Gamma(\Lambda^k M) \to C^k_\Delta(M;\mathbf{R}) \cong (C_{k,\Delta}(M;\mathbf{R}))^*$ by

$$\mathcal{I}(\omega)(c) = \int_c \omega$$

   for $\omega \in \Gamma(\Lambda^k M)$ and $c \in C_{k,\Delta}(M;\mathbf{R})$.
   (a) Show that $\mathcal{I}(d\omega) = \partial^*\mathcal{I}(\omega)$.
   (b) Show that $\mathcal{I}$ induces a linear map

$$\mathcal{I}_*\colon H^k_{dR}(M) \to H^k_\Delta(M;\mathbf{R}) \cong (H_{k,\Delta}(M;\mathbf{R}))^*.$$

   [See Problem 6 for more on this map.]
4. Let $\mathcal{F} \to M$ be a sheaf, and $\mathcal{U},\mathcal{V}$ open covers of M such that $\mathcal{V}$ refines $\mathcal{U}$. A *refining map* is a map $\rho\colon \mathcal{V} \to \mathcal{U}$ such that $V \subset \rho V$ for every $V \in \mathcal{V}$. For any refining map ρ, define the induced cochain map $\rho^\#\colon \mathcal{C}^k(\mathcal{U};\mathcal{F}) \to \mathcal{C}^k(\mathcal{V};\mathcal{F})$ by

$$(\rho^\# c)(V_0,\dots,V_k) = c(\rho V_0,\dots,\rho V_k),$$

   and the induced cohomology map $\rho^*\colon H^k(\mathcal{U};\mathcal{F}) \to H^k(\mathcal{V};\mathcal{F})$ by $\rho^*[c] = [\rho^\# c]$. Prove that ρ^* depends only on the covers $\mathcal{U}$ and $\mathcal{V}$, and not on the refining map ρ, as follows.

(a) Given two refining maps $\rho, \tilde{\rho}$, define a map $\theta\colon \mathcal{C}^k(\mathfrak{U};\mathcal{F}) \to \mathcal{C}^{k-1}(\mathcal{V};\mathcal{F})$ by

$$(\theta c)(V_0, \dots, V_{k-1}) = \sum_{i=0}^{k-1}(-1)^i c(\rho V_0, \dots, \rho V_i, \tilde{\rho} V_i, \dots, \tilde{\rho} V_{k-1}).$$

Show that $\theta\delta + \delta\theta = \rho^{\#} - \tilde{\rho}^{\#}$.

(b) Show that $\rho^* = \tilde{\rho}^*$.

5. Let M be a smooth manifold, and let $\mathcal{E}^0$ denote the sheaf of germs of C^∞ functions on M. For any open subset $U \subset M$, show that the space of sections $\mathcal{E}^0(U)$ is isomorphic to $C^\infty(U)$ via the map $f \leftrightarrow \sigma_f$, where

$$\sigma_f(x) = [f]_x.$$

III. Optional problems.

6. [HARD] Prove that the map $\mathcal{I}_*$ defined in Problem 3 actually realizes the deRham isomorphism. [Hint: you might try to trace through the isomorphisms of the deRham theorem and show this directly. See also Chapter 5 of *Foundations of Differentiable Manifolds and Lie Groups* by F. Warner for another approach.]

7. In some treatments of sheaf theory, the spaces of sections $\mathcal{F}(U)$ are treated as the primary objects instead of the topological space $\mathcal{F}$. This problem outlines that approach.

   Define a *presheaf* on a manifold M as an assignment of an abelian group $\mathcal{F}(U)$ to each open subset $U \subset M$, together with homomorphisms (called "restriction maps") $\rho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ whenever $V \subset U$, satisfying $\rho_{UU} = id_U$ and $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ whenever $W \subset V \subset U$.

   (a) Show that every sheaf gives rise to a presheaf in a canonical way.

   (b) By mimicking the construction of the sheaf of germs of C^∞ functions, show that every presheaf gives rise to a sheaf such that elements of $\mathcal{F}(U)$ can naturally be identified with sections of $\mathcal{F}$ over U. (This operation is sometimes given the unwieldy name of "sheafification functor".)

   (c) Suppose $\mathcal{F}$ is a sheaf. Show that the sheafification of the presheaf associated with $\mathcal{F}$ is canonically isomorphic to $\mathcal{F}$.

   (d) Show that the converse is not true, that is, there are presheaves that are not equal to the spaces of sections of their sheafifications. Formulate necessary and sufficient conditions on a presheaf so that equality will hold. (Presheaves satisfying these conditions are called "complete".)

   [See the book by Warner cited in the preceding problem for more.]