Math 548 | **Geometric Structures** | **Winter 1995**

ASSIGNMENT #3

DUE WEDNESDAY, 3/1/95

Reading: do Carmo, Chapter 13.

I. Practice problems.

1. Prove *Taylor's formula with integral remainder* for a smooth function f defined in a neighborhood of $0 \in \mathbf{R}^n$:

$$f(x) = P_k(x) + \sum_{i_1,\dots,i_{k+1}=1}^{n} \int_0^1 \frac{(1-t)^k}{k!} \frac{\partial^{k+1} f(tx)}{\partial x^1 \dots \partial x^{k+1}}\, dt,$$

where P_k is the k-th degree Taylor polynomial of f at 0. As a corollary, deduce that if 0 is a critical point of f, then f can be written in a neighborhood of 0 as

$$f(x) = f(0) + \sum_{i,j=1}^{n} x^i x^j h_{ij}(x)$$

for some smooth functions h_{ij}.

II. Required problems.

2. Let M be a Riemannian manifold whose sectional curvatures satisfy $K \le 1/R^2$. Suppose $r < \min\{\frac{1}{2}\pi R, \frac{1}{2} i(M)\}$. Show that any closed geodesic ball of radius r is strictly convex.

[Hint: Show that, if the result fails, there is a geodesic segment γ in $\overline{B_r}(p)$ which is tangent to $S_r(p)$ at an interior point; by reparametrizing γ, we can assume the point of tangency is $\gamma(0)$. Let $\sigma_s\colon [0,a] \to M$ be the minimizing geodesic from p to $\gamma(s)$. Using the second variation formula, show that

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(\sigma_s) = I(J,J)$$

where J is the Jacobi field $J(t) = \partial\sigma_s(t)/\partial s|_{s=0}$ along σ_0. Then, using an exponential correspondence with the sphere of radius R and the Index Lemma as in the proof of the Rauch comparison theorem, show that $I(J,J) > 0$, and derive a contradiction.]

3. Let M and N be compact manifolds, and suppose they are *cobordant*, i.e. there exists a smooth, compact manifold-with-boundary B whose boundary is (diffeomorphic to) the disjoint union of M and N. Show that, up to homeomorphism, N can be obtained by applying finitely many surgeries to M. [A *surgery* on M is obtained by removing the interior of an embedded $S^k \times D^{n-k}$ and gluing in $D^{k+1} \times S^{n-k-1}$ in its place, noting that $\partial(S^k \times D^{n-k}) = \partial(D^{k+1} \times S^{n-k-1}) = S^k \times S^{n-k-1}$. You may assume without proof the existence of a Morse function f on B such that $f = 0$ on M and $f = 1$ on N, and such that all the critical points of f occur in the interior of B.]

4. Let S be a compact 2-manifold, and suppose S admits a Morse function with exactly n critical points. What can you say about the topology of S?

III. Optional problems.

5. do Carmo p. 188 #11, 12. (These problems construct the *complex projective spaces*, which show that the sphere theorem is sharp in even dimensions: they are compact, simply-connected manifolds with sectional curvatures satisfying $\frac{1}{4} \leq K \leq 1$ at each point.) You might have to read (and do) parts or all of problems 8–10 also.