Math 548 **Geometric Structures** **Winter 1995**

ASSIGNMENT #1

DUE 2/10/95

Reading: do Carmo, Chapters 10 and 11.

I. Practice problems.

1. Let $\gamma\colon [0,a] \to M$ be a geodesic segment. The *Jacobi boundary problem* is, given vectors $V \in T_{\gamma(0)}M$ and $W \in T_{\gamma(a)}M$, to find a Jacobi field J along γ such that $J(0) = V$ and $J(a) = W$. It is *well posed* if there is a unique solution for every choice of V, W. Show that the Jacobi boundary problem is well posed if and only if $\gamma(0)$ and $\gamma(a)$ are not conjugate along γ.
2. do Carmo p. 250 #1.

II. Required problems.

3. do Carmo p. 235 #1.
4. do Carmo p. 250 #2:
   (a) Prove Wirtinger's inequality using do Carmo's hint.
   (b) Give another, more direct, proof by expanding the quantity $\int_0^\pi g'(t)^2 \sin^2 t\, dt$, where $g(t) = f(t)/(\sin t)$, and integrating by parts.
5. do Carmo p. 252 #6.
6. Let $\gamma\colon [0,a] \to M$ be a geodesic segment. For $0 < \tau \le a$, let $\mathcal{V}_\tau$ denote the space of piecewise smooth vector fields along $\gamma[0,\tau]$ which vanish at 0 and τ; let I_τ denote the index form on $\mathcal{V}_\tau$; and let $i(\tau)$ denote the index of I_τ. Show that i is lower semicontinuous. More precisely, for any $\tau \in (0,a]$, show that there exists $\varepsilon > 0$ such that i is constant on $(\tau - \varepsilon, \tau]$.

   [Hint: Choose $0 = t_0 < t_1 < \cdots < t_{k+1} = a$ such that the restriction of γ to $[t_{i-1}, t_i]$ is minimizing, as in the proof of the Index Theorem. Let $\mathcal{V}_\tau^- \subset \mathcal{V}_\tau$ denote the (finite-dimensional) space of broken Jacobi fields that vanish at 0 and τ, with breaks only at the points t_i. As long as $t_m < \tau \le t_{m+1}$, each $\mathcal{V}_\tau^-$ is isomorphic to the fixed finite-dimensional space $\mathcal{T}_m = T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_m)}M$. By means of an explicit isomorphism between $\mathcal{V}_\tau^-$ and this space, reduce I_τ to a (varying) bilinear form on $\mathcal{T}_m$, and show that this form varies continuously as long as $\tau \in (t_m, t_{m+1}]$. To prove that it varies continuously, it will be helpful to express Jacobi fields on $[t_m, \tau]$ in terms of a fixed basis of Jacobi fields on $[t_m, t_{m+1}]$.]

III. Optional problems.

7. Give yet a third proof of Wirtinger's inequality (Problem 4 above) using Fourier series (without looking up the proof referred to in do Carmo).
8. do Carmo p. 251 #3.